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Time-Dependent Automorphism Inducing Diffeomorphisms In Vacuum Bianchi Cosmologies And The Complete Closed Form Solutions For Type II & V

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Abstract

We investigate the set of spacetime general coordinate transformations (G.C.T.) which leave the line element of a generic Bianchi Type Geometry, quasi-form invariant; i.e. preserve manifest spatial Homogeneity. We find that these G.C.T.'s, induce special time-dependent automorphic changes, on the spatial scale factor matrix $\gamma_{\alpha\beta}(t)$ –along with corresponding changes on the lapse function $N(t)$ and the shift vector $N^{\alpha}(t)$. These changes, which are Bianchi Type dependent, form a group and are, in general, different from those induced by the group SAut(G)

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advocated in earlier investigations as the relevant symmetry group; they are used to simplify the form of the line element –and thus simplify Einstein's equations as well–, without losing generality. As far as this simplification procedure is concerned, the transformations found, are proved to be essentialy unique. For the case of Bianchi Types II and V, where the most general solutions are known –Taub's and Joseph's, respectively–, it is explicitly verified that our transformations and only those, suffice to reduce the generic line element, to the previously known forms. It becomes thus possible, –for these Types– to give in closed form, the most general solution, containing all the necessary "gauge" freedom.

1 Introduction

It is well known, that spatial homogeneity reduces Einstein's field equations for pure gravity, to a system of ten coupled O.D.E.'s with respect to time [1]: one equation quadratic in the velocities $\dot{\gamma}_{\alpha\beta}$ and algebraic in N^2 ($G_{00} = 0$), three linear in velocities and also algebraic in N^α ($G_{0i} = 0$), and the six spatial equations ($G_{ij} = 0$) which are linear in $\ddot{\gamma}_{\alpha\beta}$ and are also involving $N, \dot{N}, N^\alpha, \dot{N}^\alpha, \gamma_{\alpha\beta}, \dot{\gamma}_{\alpha\beta}$.

In attempting to find solutions to this set of equations, it is natural –although seldom adopted in the literature– to solve the quadratic constraint for N^2 and the linear constraint equations for as many of the N^α's as possible; then substitute into the remaining spatial equations. When this is done, the spatial equations can be solved for only 6-4=2 independent accelerations $\ddot{\gamma}_{\alpha\beta}$. Only for Bianchi Type II and III –a particular VI case– we can solve for 6-3=3 accelerations, since only two of the three linear constraints are independent; but then in both of these cases, a linear combination of the N^a's remains arbitrary and counterbalances the existence of the third independent acceleration. Thus, the general solution to the above mentioned system of equations will, in every Bianchi Type, involve four arbitrary functions of time, whose specification should, somehow, correspond to a choice of time and space coordinates –in complete analogy to the full pure gravity, whereby four arbitrary functions of the spacetime coordinates, are expected to enter the general solution.

In the literature a different approach is more frequently met. It involves an a priori gauge choice of coordinate system: As far as time is concerned, one may set N to be either an

explicit function of time –say 1 or t^2 e.t.c.–, or some combination of $\gamma_{\alpha\beta}$'s –see (2.8). For the spatial coordinates, the depicted situation is more vague. In some works, N^{α}'s are set to zero, in others, some N^{α}'s are retained. In any case, most of these choices, are considered as being, more or less, inequivalent and their connection to the well-known existence of Gauss-normal coordinates ($g_{00} = -1,\ g_{0i} = 0$) [2], is not at all clear. When such a gauge choice has been made, the spatial equations can be solved for all 6 independent $\ddot{\gamma}_{\alpha\beta}(t)$. The constraint equations become then algebraic equations, restricting the initial data –needed to specify a particular solution of the spatial equations.

In both approaches, the ensuing equations are still too difficult to handle; thus further simplifying hypotheses are employed, such as $N^{\alpha}(t) = 0$, leading to $\gamma_{\alpha\beta} = diag((a^2(t), b^2(t), c^2(t)))$ for Class A Types e.t.c. For the Bianchi types I and IX, the hypothesis $N^{\alpha}(t) = 0$ and $\gamma_{\alpha\beta} = diag((a^2(t), b^2(t), c^2(t)))$, is known to be linked to kinematics and/or dynamics –although in a, somewhat, vague way see e.g. [3] and Ryan in [1]. In all other cases, this or any other simplifying hypothesis used, is interpreted only as an ansatz to be tested at the end, i.e. after having solved all the –further simplified– equations. For example, to take an extreme case, diagonality of $\gamma_{\alpha\beta}(t)$ together with the vanishing of the shift vector is known to lead to incompatibility for Bianchi types IV, VII (Class B) [4, 5], as well as for the biaxial type VIII cases $(a^2, a^2, c^2), (a^2, b^2, a^2)$, [5]. The diversity of the various ansatzen appearing in the literature, causes a considerable degree of fragmentation.

It has long been suspected and/or known, that automorphisms, ought to play an important role in a unified treatment of this problem. The first mention, goes back

to the first of [6]. More recently, Jantzen, –second of [6]– has used Time Dependent Automorphism Matrices, as a convenient parametrization of a general positive definite 3×3 scale factor matrix $\gamma_{\alpha\beta}(t)$, in terms of a –desired– diagonal matrix. His approach, is based on the orthonormal frame bundle formalism, and the main conclusion is (third of [6], pp. 1138): " … *the special automorphism matrix group SAut(G), is the symmetry group of the ordinary differential equations, satisfied by the metric matrix* $\gamma_{\alpha\beta}$*, when no sources are present* … " Later on, Samuel and Ashtekar in [7], have seen automorphisms, as a result of general coordinate transformations. Their spacetime point of view, has led them, to consider the –so called– "Homogeneity Preserving Diffeomorphisms", and link them, to topological considerations.

In this paper, we also take a spacetime point of view, and try to avoid the fragmentation –mentioned above–, by revealing those G.C.T.'s, which enable us to simplify the line-element –and therefore Einstein's equations–, while at the same time, they preserve manifest spatial homogeneity. We are, thus, able to identify special automorphic transformations of $\gamma_{\alpha\beta}(t)$, along with corresponding –non tensorial, for the shift vector– changes of N, N^{α} which allow us to set $N^{\alpha} = 0$ and bring $\gamma_{\alpha\beta}(t)$ to some irreducible, simple –though not unique– form.

The structure of the paper, is organized as follows:

In section 2, after establishing the existence of Time-Dependent Automorphism Inducing Diffeomorphisms (A.I.D.'s), the general irreducible form of the line element for all Bianchi Types is given, and a uniqueness theorem, is proven.

In section 3, attention is focused on Bianchi Types II and V. Einstein's equations obtain-

ing from the irreducible form of the line element, are explicitly written down and completely integrated. The uniqueness of the transformations given in section 2, is explicitly verified, with the aid of the well known Taub's and Joseph's solution –respectively. As a result, we give the closed form of the most general line elements, satisfying equations (2.5).

Finally, some concluding remarks are included in the discussion.

2 Time Dependent Automorphism Inducing Diffeomorphisms

It is well known that the vacuum Einstein field equations can be derived from an action principle:

$$\mathcal{A} = \frac{-1}{16\pi} \int \sqrt{-^{(4)}g} \; ^{(4)}R \; d^4x \tag{2.1}$$

(we use geometrized units i.e. $G = c = 1$)

The standard canonical formalism [8] makes use of the lapse and shift functions appearing in the 4-metric:

$$ds^2 = (N^i N_i - N^2)dt^2 + 2N_i dx^i dt + g_{ij} dx^i dx^j \tag{2.2}$$

From this line-element the following set of equations obtains, expressed in terms of the extrinsic curvature:

$$K_{ij} = \frac{1}{2N}(N_{i|j} + N_{i|j} - \frac{\partial g_{ij}}{\partial t})$$

$$H_0 = \sqrt{g}\ (K_{ij}K^{ij} - K^2 + R) = 0 \tag{2.3a}$$

$$H_i = 2\sqrt{g}\ (K^j_{i|j} - K_{|i}) = 0 \tag{2.3b}$$

$$\begin{aligned}\frac{1}{\sqrt{g}}\frac{d}{dt}[\sqrt{g}\ (K^{ij} - Kg^{ij})] = -N(R^{ij} - \frac{1}{2}R\ g^{ij}) - \frac{N}{2}(K_{kl}K^{kl} - K^2)g^{ij} \\ +2N(K^{ik}K^j_k - K\ K^{ij}) - (N^{|ij} - N^{|l}_{|l}g^{ij}) + [(K^{ij} - K\ g^{ij})N^l]_{|l} \\ -N^i_{|l}(K^{lj} - K\ g^{lj}) - N^j_{|l}(K^{li} - K\ g^{li})\end{aligned} \tag{2.3c}$$

This set is equivalent to the ten Einstein's equations.

In cosmology, we are interested in the class of spatially homogeneous spacetimes, characterized by the existence of an m-dimensional isometry group of motions G, acting transitively on each surface of simultaneity Σ_t. When m is greater than 3 and there is no proper invariant subgroup of dimension 3, the spacetime is of the Kantowski-Sachs type [9] and will not concern us further. When m equals the dimension of Σ_t –which is 3–, there exist 3 basis one-forms σ^α_i satisfying:

$$d\sigma^\alpha = C^\alpha_{\beta\gamma}\ \sigma^\beta \wedge \sigma^\gamma \Leftrightarrow \sigma^\alpha_{i,\ j} - \sigma^\alpha_{j,\ i} = 2C^\alpha_{\beta\gamma}\ \sigma^\gamma_i\ \sigma^\beta_j \tag{2.4a}$$

where $C^\alpha_{\beta\gamma}$ are the structure constants of the corresponding isometry group.

In this case there are local coordinates $t,\ x^i$ such that the line element in (2.2) assumes the form:

$$\begin{aligned}ds^2 = (N^\alpha(t)N_\alpha(t) - N^2(t))dt^2 + 2N_\alpha(t)\sigma^\alpha_i(x)dx^i dt \\ + \gamma_{\alpha\beta}\sigma^\alpha_i(x)\sigma^\beta_j(x)dx^i dx^j\end{aligned} \tag{2.4b}$$

Latin indices, are spatial with range from 1 to 3. Greek indices, number the different basis 1-forms, take values in the same range, and are lowered and raised by $\gamma_{\alpha\beta}$, and $\gamma^{\alpha\beta}$ respectively.

A commitment concerning the topology of the 3-surface, is pertinent here, especially in view of the fact that we wish to consider diffeomorphisms [7]; we thus assume that G is simply connected and the 3-surface Σ_t can be identified with G, by singling out a point p of Σ_t, as the identity e, of G.

If we insert relations (2.4) into equations (2.3), we get the following set of ordinary differential equations for the Bianchi-Type spatially homogeneous spacetimes:

$$E_0 \doteq K^{\alpha}_{\beta}\ K^{\beta}_{\alpha} - K^2 + R = 0 \tag{2.5a}$$

$$E_\alpha \doteq K^{\mu}_{\alpha}\ C^{\epsilon}_{\mu\epsilon} - K^{\mu}_{\epsilon}\ C^{\epsilon}_{\alpha\mu} = 0 \tag{2.5b}$$

$$E^{\alpha}_{\beta} \doteq \dot{K}^{\alpha}_{\beta} - NKK^{\alpha}_{\beta} + NR^{\alpha}_{\beta} + 2N^{\rho}(K^{\alpha}_{\nu}\ C^{\nu}_{\beta\rho} - K^{\nu}_{\beta}\ C^{\alpha}_{\nu\rho}) \tag{2.5c}$$

where $K^{\alpha}_{\beta} = \gamma^{\alpha\rho}K_{\rho\beta}$ and

$$K_{\alpha\beta} = -\frac{1}{2N}(\dot{\gamma}_{\alpha\beta} + 2\gamma_{\alpha\nu}C^{\nu}_{\beta\rho}N^{\rho} + 2\gamma_{\beta\nu}C^{\nu}_{\alpha\rho}N^{\rho}) \tag{2.6}$$

$$\begin{aligned} R_{\alpha\beta} = {} & C^{\kappa}_{\sigma\tau}C^{\lambda}_{\mu\nu}\gamma_{\alpha\kappa}\gamma_{\beta\lambda}\gamma^{\sigma\nu}\gamma^{\tau\mu} + 2C^{\lambda}_{\alpha\kappa}C^{\kappa}_{\beta\lambda} + 2C^{\mu}_{\alpha\kappa}C^{\nu}_{\beta\lambda}\gamma_{\mu\nu}\gamma^{\kappa\lambda} \\ & + 2C^{\lambda}_{\beta\kappa}C^{\mu}_{\mu\nu}\gamma_{\alpha\lambda}\gamma^{\kappa\nu} + 2C^{\lambda}_{\alpha\kappa}C^{\mu}_{\mu\nu}\gamma_{\beta\lambda}\gamma^{\kappa\nu} \end{aligned} \tag{2.7}$$

When $N^{\alpha} = 0$, equation (2.5c) reduces to the form of the equation given in [10]. Equation set (2.5), forms what is known as a –complete– perfect ideal; that is, there are no

integrability conditions obtained from this system. So, with the help of (2.5c), (2.6), (2.7), it can explicitly be shown, that the time derivatives of (2.5a) and (2.5b) vanish identically. The calculation is staightforward –although somewhat lengthy. It makes use of the Jacobi identity $C^{\alpha}_{\rho\beta}C^{\rho}_{\gamma\delta}+C^{\alpha}_{\rho\delta}C^{\rho}_{\beta\gamma}+C^{\alpha}_{\rho\gamma}C^{\rho}_{\delta\beta}=0$, and its contracted form $C^{\alpha}_{\alpha\beta}C^{\beta}_{\gamma\delta}=0$.

The vanishing of the derivatives of the 4 constrained equations:

$E_0 = 0, E_\alpha = 0$, implies that these equations, are first integrals of equations (2.5c) –moreover, with vanishing integration constants. Indeed, algebraically solving (2.5a), (2.5b) for $N(t), N^\alpha(t)$, respectively and substituting in (2.5c), one finds that in all –but Type II and III– Bianchi Types, equations (2.5c), can be solved for only 2 of the 6 accelerations $\ddot{\gamma}_{\alpha\beta}$ present. In Type II and III, the independent accelerations are 3, since E_α are not independent and thus, can be solved for only 2 of the 3 N^α's. But then in both of these cases, a linear combination of the N^a's remains arbitrary, and counterbalances the extra independent acceleration. Thus, in all Bianchi Types, 4 arbitrary functions of time enter the general solution to the set of equations (2.5). Based on the intuition gained from the full theory, one could expect this fact to be a reflection of the only known covariance of the theory; i.e. of the freedom to make arbitrary changes of the time and space coordinates.

The rest of this section is devoted to the investigation of the existence, uniqueness, and properties of general coordinate transformations –containing 4 arbitrary functions of time–, which on the one hand, must preserve the manifest spatial homogeneity, of the line element (2.4b), and on the other hand, must be symmetries of equations (2.5).

As far as time reparametrization is concerned the situation is pretty clear: If a transfor-

mation

$$t \to \tilde{t} = g(t) \Leftrightarrow t = f(\tilde{t}) \tag{2.8a}$$

is inserted in the line element (2.4b), it is easily inferred that

$$\gamma_{\alpha\beta}(t) \to \gamma_{\alpha\beta}(f(\tilde{t})) \equiv \tilde{\gamma}_{\alpha\beta}(\tilde{t}) \tag{2.8b}$$

$$\begin{aligned} N(t) &\to \pm\, N(f(\tilde{t}))\frac{df(\tilde{t})}{d\tilde{t}} \equiv \widetilde{N}(\tilde{t}) \\ N^{\alpha}(t) &\to N^{\alpha}(f(\tilde{t}))\frac{df(\tilde{t})}{d\tilde{t}} \equiv \widetilde{N}^{\alpha}(\tilde{t}) \end{aligned} \tag{2.8c}$$

Accordingly, K^{α}_{β} transforms under (2.8a) as a scalar and thus (2.5a), (2.5b) are also scalar equations while (2.5c) gets multiplied by a factor $df(\tilde{t})/d\tilde{t}$. Thus, given a particular solution to equations (2.5), one can always obtain an equivalent solution, by arbitrarily redefining time. Hence, we understand the existence of one arbitrary function of time in the general solution to Einstein's equations (2.5). In order to understand the presence of the rest 3 arbitrary functions of time it is natural to turn our attention to the tranformations of the 3 spatial coordinates x^i. To begin with, consider the transformation:

$$\begin{aligned} \tilde{t} &= t \Leftrightarrow t = \tilde{t} \\ \tilde{x}^i &= g^i(x^j, t) \Leftrightarrow x^i = f^i(\tilde{x}^j, \tilde{t}) \end{aligned} \tag{2.9}$$

It is here understood, that our previous assumption concerning the topology of G and the identification of Σ_t with G, is valid for all values of the parameter t, for which the transformation is to be well defined.

Under these transformations, the line element (2.4b) becomes:

$$
\begin{aligned}
ds^2 = &[(N^\alpha N_\alpha - N^2) + \frac{\partial f^i}{\partial \tilde{t}} \frac{\partial f^j}{\partial \tilde{t}} \ \sigma_i^\alpha(f) \sigma_j^\beta(f) \gamma_{\alpha\beta}(\tilde{t}) \\
&+ 2\sigma_i^\alpha(f) \frac{\partial f^i}{\partial \tilde{t}} N_\alpha(\tilde{t})] d\tilde{t}^2 \\
&+ 2\sigma_i^\alpha(x) \frac{\partial x^i}{\partial \tilde{x}^m} [N_\alpha(\tilde{t}) + \sigma_j^\beta(x) \frac{\partial x^j}{\partial \tilde{t}} \gamma_{\alpha\beta}(\tilde{t})] d\tilde{x}^m d\tilde{t} \\
&+ \sigma_i^\alpha(x) \sigma_j^\beta(x) \gamma_{\alpha\beta}(\tilde{t}) \frac{\partial x^i}{\partial \tilde{x}^m} \frac{\partial x^j}{\partial \tilde{x}^n} \ d\tilde{x}^m d\tilde{x}^n
\end{aligned}
\tag{2.10}
$$

Since our aim, is to retain manifest spatial homogeneity of the line element (2.4b), we have to refer the form of the line element given in (2.10) to the old basis $\sigma_i^\alpha(\tilde{x})$ at the new spatial point $\tilde{x}^i$. Since σ_i^α –both at x^i and $\tilde{x}^i$–, as well as, $\partial x^i / \partial \tilde{x}^j$, are invertible matrices, there always exists a non-singular matrix $\Lambda_\mu^\alpha(\tilde{x}, \tilde{t})$ and a triplet $P^\alpha(\tilde{x}, \tilde{t})$, such that:

$$
\begin{aligned}
\sigma_i^\alpha(x) \frac{\partial x^i}{\partial \tilde{x}^m} =& \Lambda_\mu^\alpha(\tilde{x}, \tilde{t}) \sigma_m^\mu(\tilde{x}) \\
\sigma_i^\alpha(x) \frac{\partial x^i}{\partial \tilde{t}} =& P^\alpha(\tilde{x}, \tilde{t})
\end{aligned}
\tag{2.11}
$$

The above relations, must be regarded as definitions, for the matrix Λ_μ^α and the triplet P^α. With these identifications the line element (2.10) assumes the form:

$$
\begin{aligned}
ds^2 = &[(N^\alpha N_\alpha - N^2) + P^\alpha(\tilde{x}, \tilde{t}) P^\beta(\tilde{x}, \tilde{t}) \gamma_{\alpha\beta}(\tilde{t}) + 2P^\alpha(\tilde{x}, \tilde{t}) N_\alpha(\tilde{t})] d\tilde{t}^2 \\
&+ 2\Lambda_\mu^\alpha(\tilde{x}, \tilde{t}) \sigma_m^\mu(\tilde{x}) [N_\alpha(\tilde{t}) + P^\beta(\tilde{x}, \tilde{t}) \gamma_{\alpha\beta}(\tilde{t})] d\tilde{x}^m d\tilde{t} \\
&+ \Lambda_\mu^\alpha(\tilde{x}, \tilde{t}) \Lambda_\nu^\beta(\tilde{x}, \tilde{t}) \gamma_{\alpha\beta}(\tilde{t}) \sigma_m^\mu(\tilde{x}) \sigma_n^\nu(\tilde{x}) d\tilde{x}^m d\tilde{x}^n
\end{aligned}
\tag{2.12}
$$

If, following the spirit of [7], we wish the transformation (2.9) to be manifest homogeneity preserving i.e. to have a well defined, non-trivial action on $\gamma_{\alpha\beta}(t)$, $N(t)$ and $N^\alpha(t)$, we must impose the condition that $\Lambda_\mu^\alpha(\tilde{x}, \tilde{t})$ and $P^\alpha(\tilde{x}, \tilde{t})$ do not depend on the spatial point

$\tilde{x}$, i.e. $\Lambda^\alpha_\mu = \Lambda^\alpha_\mu(\tilde{t})$ and $P^\alpha = P^\alpha(\tilde{t})$. Then (2.12) is written as:

$$
\begin{aligned}
ds^2 &= [(N^\alpha N_\alpha - N^2) + P^\alpha P^\beta \gamma_{\alpha\beta} + 2P^\alpha N_\alpha] d\tilde{t}^2 \\
&+ 2\Lambda^\alpha_\mu \sigma^\mu_m(\tilde{x})[N_\alpha + P^\beta \gamma_{\alpha\beta}] d\tilde{x}^m d\tilde{t} \\
&+ \Lambda^\alpha_\mu \Lambda^\beta_\nu \gamma_{\alpha\beta} \sigma^\mu_m(\tilde{x}) \sigma^\nu_n(\tilde{x}) d\tilde{x}^m d\tilde{x}^n \Rightarrow \\
ds^2 &\equiv (\widetilde{N}^\alpha \widetilde{N}_\alpha - \widetilde{N}^2) d\tilde{t}^2 + 2\widetilde{N}_\alpha(\tilde{t}) \sigma^\alpha_i(\tilde{x}) d\tilde{x}^i d\tilde{t} \\
&+ \widetilde{\gamma}_{\alpha\beta}(\tilde{t}) \sigma^\alpha_i(\tilde{x}) \sigma^\beta_j(\tilde{x}) d\tilde{x}^i d\tilde{x}^j
\end{aligned} \tag{2.13}
$$

with the allocations:

$$\widetilde{\gamma}_{\alpha\beta} = \Lambda^\mu_\alpha \Lambda^\nu_\beta \gamma_{\mu\nu} \tag{2.14a}$$

$$\widetilde{N}_\alpha = \Lambda^\beta_\alpha (N_\beta + P^\rho \gamma_{\rho\beta}) \ \ \textit{and} \ \ \textit{thus} \ \ \widetilde{N}^\alpha = S^\alpha_\beta (N^\beta + P^\beta) \tag{2.14b}$$

$$\widetilde{N} = N \tag{2.14c}$$

(where $S = \Lambda^{-1}$).

Of course, the demand that Λ^α_β and P^α must not depend on the spatial point $\tilde{x}^i$, changes the character of (2.11), from identities, to the following set of differential restrictions on the functions defining the transformation:

$$\frac{\partial f^i}{\partial \tilde{x}^m} = \sigma^i_\alpha(f) \Lambda^\alpha_\beta(\tilde{t}) \sigma^\beta_m(\tilde{x}) \tag{2.15a}$$

$$\frac{\partial f^i}{\partial \tilde{t}} = \sigma^i_\alpha(f) P^\alpha(\tilde{t}) \tag{2.15b}$$

Equations (2.15) constitute a set of first-order highly non-linear P.D.E.'s for the unknown functions f^i. The existence of local solutions to these equations is guaranteed by

Frobenius theorem [11] as long as the necessary and sufficient conditions:

$$\frac{\partial}{\partial\tilde{x}^j}\Big(\frac{\partial f^i}{\partial\tilde{x}^m}\Big)-\frac{\partial}{\partial\tilde{x}^m}\Big(\frac{\partial f^i}{\partial\tilde{x}^j}\Big)=0$$

$$\frac{\partial}{\partial\tilde{t}}\Big(\frac{\partial f^i}{\partial\tilde{x}^m}\Big)-\frac{\partial}{\partial\tilde{x}^m}\Big(\frac{\partial f^i}{\partial\tilde{t}}\Big)=0$$

hold. Through (2.15) and repeated use of (2.4a), these equations reduce respectively to:

$$\Lambda^\alpha_\mu C^\mu_{\beta\gamma}=\Lambda^\rho_\beta\Lambda^\sigma_\gamma C^\alpha_{\rho\sigma} \tag{2.16}$$

$$P^\mu C^\alpha_{\mu\nu}\Lambda^\nu_\beta=\frac{1}{2}\dot{\Lambda}^\alpha_\beta \tag{2.17}$$

It is noteworthy that the solutions to (2.16) and (2.17), –by virtue of (2.14)– form a group, with composition law:

$$(\Lambda_3)^\alpha_\beta=(\Lambda_1)^\alpha_\varrho(\Lambda_2)^\varrho_\beta$$

$$(P_3)^a=(\Lambda_1)^\alpha_\beta(P_2)^\beta+(P_1)^a$$

where $(\Lambda_1,\ P_1)$ and $(\Lambda_2,\ P_2)$, are two successive transformations of the form (2.14). Note also, that a constant automorphism is always a solution of (2.16), (2.17); indeed, $\Lambda^a_\beta(t)=\Lambda^a_\beta$ and $P^a(t)=0$ solve these equations. Thus, Λ^a_β and $P^a=0$ can be regarded as the remaining gauge symmetry, after one has fully used the arbitrary functions of time, appearing in a solution $\Lambda^a_\beta(t)$ and $P^a(t)$. Consequently one can, at first sight, regard all the arbitrary constants encountered when integrating (2.17), as absorbable in the shift, since the transformation law for the shift, is then tensorial. This is certainly true, as long

as there is a non zero initial shift. However, if one has used the independent functions of time, in order to set the shift zero, then the constants remaining within Λ^a_β, are not absorbable. It is this kind of constants that we explicitly present below, when we give the solutions to (2.16), (2.17) for all Bianchi Types. A relevant nice discussion, distinguishing between genuine gauge symmetries (cf. arbitrary functions of time) and rigid symmetries (cf. arbitrary constants), is presented in [12]. There a different definition of manifest homogeneity preserving diffeomorphisms –stronger than the one adopted in this work– is used, and results in only the inner automorphisms being allowed to acquire t dependence. In connection to this, it is interesting to observe that (2.16-17) give essentially the same results: notice that $2P^\mu C^\alpha_{\mu\beta}$ is, by definition, the generator of Inner Automorphisms. Thus there is always a $\lambda^\alpha_\beta(t) \equiv Exp(2P^\mu C^\alpha_{\mu\beta}) \in \mathrm{IAut(G)}$ satisfying (2.17). If we now parameterize the general solution to (2.16-17) by $\Lambda^\alpha_\beta(t) = \lambda^\alpha_\varrho(t) U^\varrho_\beta(t)$ and substitute in these relations, we deduce that the matrix U is a constant automorphism. This analysis is verified in the explicit solutions to (2.16-17), presented latter.

Equation (2.16) is satisfied if and only if, $\Lambda^\alpha_\beta(t)$ is an element of the automorphism group of the Lie algebra determined by the $C^\alpha_{\beta\gamma}$. Equation (2.17) further restricts the form of $\Lambda^\alpha_\beta(t)$ and $P^\alpha(t)$, so that manifest spatial homogeneity is preserved despite the mixing of the old time and space coordinates in the new spatial coordinates $\tilde{x}^i$. Thus, it is appropriate to call transformations (2.9), satisfying conditions (2.15), (2.16), (2.17), **Time-Dependent Automorphism Inducing Diffeomorphisms**. The importance of automorphisms in Bianchi Cosmologies, has been stressed in [6]. The symmetry group of the differential equations, satisfied by $\gamma_{\alpha\beta}(t)$, –advocated in these works of Jantzen *et al*–

is the unimodular matrices SAut(G). As we shall later see, we find another symmetry group, whose time-dependent part lies essentially in IAut(G) and thus coincides with SAut(G), only for Class A Bianchi Types VI, VII, VIII, IX.

At this point it is natural to ask how this difference occurs. It is our opinion that the key elements on which the difference in the symmetry groups found rests, are:

a) The inhomogeneous transformation law (2.14b) for the shift. Indeed, Jantzen (1979), having adopted an orthonormal frame-bundle point of view, naturally assumes as his "gauge" transformation laws (2.14a,c) and the tensorial law $\bar{N}^a = S^a_\beta N^\beta$, for the shift p. 221.

b) The different definition and/or role reserved for the triplet $P^a(t)$; we define it as a sort of "velocity" of the transformation (2.9) in (2.15b) and use it in the inhomogeneous law (2.14b). On the other hand, Jantzen (1979), (p. 221) uses the corresponding quantity $\omega^a(t)$ (so called velocity of the automorphism frame)to define a new time derivative $\partial/\partial\bar{t} = \partial/\partial t + \omega^a(t)\sigma^i_a(x)\partial/\partial x^i$.

c) We concentrate on the symmetries of the O.D.E.s (2.5), i.e. of *Eistein's equations written in the invariant base*, while Jantzen, as far as we understand, focuses on the symmetries of the P.D.E.s (2.3), i.e. *of Einstein's equations, written in an arbitrary frame.*

In [7], the so called Homogeneity Preserving Diffeomorphisms, are considered in relation to the topology of Σ_t. A time-independent version of (2.15), appears in [13], where all homogeneous three-geometries, are classified in equivalence classes, with re-

spect to these "frozen" transformations. It is straightforward to check, that $E_0,\ E_\alpha,\ E^\alpha_\beta$ transform –under (2.14)– as follows:

$$\widetilde{E}_0 = E_0, \quad \widetilde{E}_\alpha = \Lambda^\beta_\alpha E_\beta, \quad \widetilde{E}^\alpha_\beta = S^\alpha_\mu \Lambda^\nu_\beta E^\mu_\nu \tag{2.18}$$

This fact, establishes the covariance of equations (2.5), under the "gauge" transformation (2.14), and implies that if $(N,\ N^\alpha,\ \gamma_{\alpha\beta})$ is a solution to Einstein's equations, so will be the set $(\widetilde{N},\ \widetilde{N}^\alpha,\ \widetilde{\gamma}_{\alpha\beta})$ –provided that, (2.16), (2.17) hold–; in fact, as the preceding exposition proves, they will be the same equations expressed in different space-time coordinate systems. Out of the twelve quantities $\Lambda^\alpha_\beta(t)$ and $P^\alpha(t)$, conditions (2.16), (2.17) leave us, as we are going to see, in every Bianchi Type, with 3 arbitrary functions of time. This fact, along with the time reparametrization covariance, completes our understanding of why four arbitrary functions of time enter the general solution to (2.5). Consequently, transformation (2.14), gives us the possibility to simplify the form of the line element, and thus of Einstein's equations without loss of generality. It is obvious, that the simplification obtained, is different for different Bianchi Types, and even within a particular Bianchi Type it is not unique –since one may "spend" the freedom of the three arbitrary functions in different ways.

A particularly interesting result, is that the shift vector $\widetilde{N}^\alpha$ can always be put to zero –perhaps at the expense of a more complicated $\widetilde{\gamma}_{\alpha\beta}$. For the sake of completeness, we give below, a detailed analysis of the space of solutions to (2.16) and (2.17), for each and every Bianchi Type (solutions to (2.16), have been presented in [14]).

To this end, recall that in 3 dimensions, the tensor $C^\alpha_{\beta\gamma}$, admits a unique decomposition

in terms of a contravariant symmetric tensor density of weight -1, $m^{\alpha\beta}$ and a covariant vector $\nu_\alpha = \frac{1}{2}C^\rho_{\alpha\rho}$ as follows [15]:

$$C^\alpha_{\beta\gamma} = m^{\alpha\delta}\varepsilon_{\delta\beta\gamma} + \nu_\beta\delta^\alpha_\gamma - \nu_\gamma\delta^\alpha_\beta$$

The contracted Jacobi identities imply that $m^{\alpha\beta}\nu_\beta = 0$, i.e. ν_α is a null eigenvector of the matrix $m^{\alpha\beta}$. Under a $GL(3,\Re)$ linear mixing of the basis 1-forms $\sigma^\alpha \to \widetilde{\sigma}^\alpha = S^\alpha_\beta\sigma^\beta$, the structure constant tensor transforms as:

$$C^\alpha_{\beta\gamma} \to \widetilde{C}^\alpha_{\beta\gamma} = S^\alpha_\mu\Lambda^\nu_\beta\Lambda^\rho_\gamma C^\mu_{\nu\rho}$$

Accordingly, the $m^{\alpha\beta}$ and ν_α transform as:

$$\widetilde{m}^{\alpha\beta} = |S|^{-1}S^\alpha_\gamma S^\beta_\delta m^{\gamma\delta}$$

$$\widetilde{\nu}_\alpha = \Lambda^\beta_\alpha\nu_\beta$$

Λ (and thus S) is called a Lie algebra automorphism if $C^\alpha_{\beta\gamma} = \widetilde{C}^\alpha_{\beta\gamma}$, i.e. if $\widetilde{m}^{\alpha\beta}$ and $\widetilde{\nu}_\alpha$ are equal to $m^{\alpha\beta}$ and ν_α respectively. In this case the automorphism conditions become:

$$m^{\alpha\beta} = |S|^{-1}S^\alpha_\gamma S^\beta_\delta m^{\gamma\delta} \tag{2.19a}$$

$$\nu_\alpha = \Lambda^\beta_\alpha\nu_\beta \tag{2.19b}$$

The different Bianchi Types, arise according to the rank and signature of $m^{\alpha\beta}$, and the vanishing or not, of ν_α. Using (2.19), one can –straightforwardly– solve the system of equations (2.16) and (2.17). We now present, the form of $\Lambda^\alpha_\beta(t)$ and $P^\alpha(t)$ satisfying (2.16), (2.17) as well as some irreducible form for $\gamma_{\alpha\beta}$, for each Bianchi Type:

Type I: $m^{\alpha\beta}=0, \nu_\alpha=0$. This Type has been exhaustively treated, in the literature ([3], [7]). We only note that –since all $C^\alpha_{\beta\gamma}$ are zero– (2.17), implies that $P^\alpha(t)$ is arbitrary and $\Lambda^\alpha_\beta(t)$ is constant. Then, (2.16) implies that Λ^α_β is an element of $GL(3,\Re)$. Thus, without loss of generality, one can set $N^\alpha=0$, –using (2.14b). A first integral of equations (2.5c) is then, $\gamma^{\alpha\rho}\dot{\gamma}_{\rho\beta}=\vartheta^\alpha_\beta$ where ϑ^α_β, is an arbitrary constant matrix. From this point, the standard textbooks, [3] deduce (using algebraic arguments) a diagonal form:

$\gamma_{\alpha\beta}=diag(e^{\alpha t}, e^{\beta t}, e^{\gamma t})$ and then using Einstein's equations find the general solution, which depends on 1 essential parameter, as expected –see table.

Indeed, from (2.5c), one has 12 initial constants; 6 $\gamma_{\alpha\beta}$, and 6 $\dot{\gamma}_{\alpha\beta}$ at some t_0 –according to Peano's theorem. The quadratic constraint equation (2.5a), reduces them to 10, and then, with the usage of constant automorphisms –which contain 9 Λ^α_β's–, the number of the remaining essential constants (or essential parameters), is $10-9=1$.

Type II rank(m)=1 and $\nu_\alpha=0$. Then, matrix $m^{\alpha\beta}$, can be cast to the form $m^{\alpha\beta}=diag(1/2,0,0)$. Equations (2.16), (2.17) give the following form for $\Lambda^\alpha_\beta(t)$:

$$\Lambda^\alpha_\beta(t)=\begin{pmatrix} \varrho_1\varrho_4-\varrho_2\varrho_3 & x(t) & y(t) \\ 0 & \varrho_1 & \varrho_2 \\ 0 & \varrho_3 & \varrho_4 \end{pmatrix}, \quad (\varrho_1,\varrho_2,\varrho_3,\varrho_4 \ \text{constants})$$

The triplet $P^\alpha(t)$ assumes the form:

$$P^\alpha(t)=(P(t),\ \frac{\varrho_1\dot{y}-\varrho_2\dot{x}}{\varrho_1\varrho_4-\varrho_2\varrho_3},\ \frac{\varrho_3\dot{y}-\varrho_4\dot{x}}{\varrho_1\varrho_4-\varrho_2\varrho_3})$$

The general solution to this Type, is Taub's solution ([16]), which contains 2 essential

parameters –see table.

Again, we can understand this number, using Peano's theorem and the arbitrary extra constants, appearing in Λ^α_β. Using $x(t)$ and $y(t)$, we start with 4 $\gamma_{\alpha\beta}$'s (i.e. we set $\gamma_{12} = \gamma_{13} = 0$) and no shift. Thus the initial arbitrary constants, are $2 \times 4 = 8$. Out of these, the quadratic constraint equation (2.5a), removes 2, and 4 more are eliminated by the 4 ϱ's, contained in Λ^α_β. So, the remaining arbitrary constants are: $8 - 2 - 4 = 2$, in accordance with the number of expected essential parameters.

Type V rank(m)=0 and $\nu_\alpha \neq 0$. Then $m^{\alpha\beta} = 0$ and we may choose $\nu_\alpha = -\frac{1}{2}\delta^3_\alpha$. Equations (2.16), (2.17) give the following form for $\Lambda^\alpha_\beta(t)$:

$$\Lambda^\alpha_\beta(t) = \begin{pmatrix} \varrho_1 P(t) & \varrho_2 P(t) & x(t) \\ \varrho_3 P(t) & \varrho_4 P(t) & y(t) \\ 0 & 0 & 1 \end{pmatrix}, \quad (\varrho_1, \varrho_2, \varrho_3, \varrho_4 \ \text{ constants})$$

with $\varrho_1\varrho_4 - \varrho_2\varrho_3 = 1$ and the triplet:

$$P^\alpha(t) = (x(\ln\frac{x}{P})^{\cdot},\ y(\ln\frac{y}{P})^{\cdot},\ (\ln\frac{1}{P})^{\cdot})$$

The general solution, is also known, as Joseph's solution ([17]), with one essential parameter.

This number comes naturally, within our method; using $x(t)$ and $y(t)$, one can eliminate γ_{13} and γ_{23}. Then, $P(t)$ can serve to set the subdeterminant of $\gamma_{\alpha\beta}$, equal to $(\gamma_{33})^2$. At this stage, we are left with 3 $\gamma_{\alpha\beta}$'s while the linear constraints equations (2.5b), imply that the shift is zero. Again, the quadratic constraint (2.5a), subtracts 2 arbitrary constants, and the constants contained in Λ^α_β, 3 more. Then, the result is: $6 - 2 - 3 = 1$,

essential constant.

Type IV rank(m)=1 and $\nu_\alpha \neq 0$. We may choose

$m^{\alpha\beta} = diag(1/2, 0, 0)$, $\nu_\alpha = -\frac{1}{2}\delta^3_\alpha$. Equations (2.16), (2.17) give the following form for $\Lambda^\alpha_\beta(t)$:

$$\Lambda^\alpha_\beta(t) = \begin{pmatrix} P(t) & P(t)\ln[\kappa P(t)] & x(t) \\ 0 & P(t) & y(t) \\ 0 & 0 & 1 \end{pmatrix}, \quad (\kappa \ \ \text{constant})$$

and the triplet:

$$P^\alpha(t) = (x(\ln\frac{x}{P})^{\cdot} - \dot{y},\ y(\ln\frac{y}{P})^{\cdot},\ (\ln\frac{1}{P})^{\cdot})$$

In this Type –which is a class B Type–, we can set $\gamma_{13} = \gamma_{23} = 0$, using $x(t)$ and $y(t)$. At this stage, the 2 of the 3 linear constraint equations, imply $N^1 = N^2 = 0$, while the third, involves $P(t)$. Thus we can further, either set $N^3 = 0$ –through (2.14b)– and retain a non-zero γ_{12}, or eliminate γ_{12}, at the expense of a non-vanishing N^3. It is well known, that $N^3 = 0$ and $\gamma_{12} = 0$, leads to incompatibility [4].

We have thus, the following counting of the essential parameters:

a) When $\gamma_{12} \neq 0$ and $N^3 = 0$, we have $8 - 2$ (from the quadratic constraint) $- 2$ (from the remaining linear equation) $- 1$ (from the constant contained in Λ^α_β) $= 3$.

b) When $\gamma_{12} = 0$ and $N^3 \neq 0$, we have $6 - 2$ (from the quadratic constraint) $- 1$ (from the constant contained in Λ^α_β) $= 3$. Notice that here, the remaining linear constraint equation, simply serves to define N^3 and thus, does not remove any constant.

Type VI (Including Type III [19], [18]) rank(m)=2, signature(m)=Lorentzian and

$\nu_\alpha \neq 0$. One convenient choice is $m^{\alpha\beta} = diag(1,-1,0)$ and $\nu_\alpha = h\delta^3_\alpha$.

Note: In the standard texts e.g. [15], the matrix $m^{a\beta}$ is given in a more complicated form, which carries part of the arbitrariness of the magnitude of the vector ν_α. In this work, we imply that $C^\alpha_{\beta\gamma}$ are given by their defining relation in terms of $\varepsilon_{\alpha\beta\gamma}$, $m^{\alpha\beta}$, ν_α.

For all values of $h \neq 0, \pm 1$, equations (2.16), (2.17) give the following form for $\Lambda^\alpha_\beta(t)$:

$$\Lambda^\alpha_\beta(t) = \begin{pmatrix} e^{-hP(t)}\lambda\cosh(P(t)) & e^{-hP(t)}\lambda\sinh(P(t)) & x(t) \\ e^{-hP(t)}\lambda\sinh(P(t)) & e^{-hP(t)}\lambda\cosh(P(t)) & y(t) \\ 0 & 0 & 1 \end{pmatrix}$$

$$(\lambda \;\; \text{constant})$$

while the triplet:

$$\begin{aligned} P^\alpha(t) =&(-\frac{(h^2-1)x(t)\dot{P}(t) + h\dot{x}(t) + \dot{y}(t)}{2(h^2-1)}, \\ &-\frac{(h^2-1)y(t)\dot{P}(t) + h\dot{y}(t) + \dot{x}(t)}{2(h^2-1)}, -\frac{\dot{P}(t)}{2}) \end{aligned}$$

For $h = 0$, –class A–, there are two solutions:

$$\Lambda^\alpha_\beta(t) = \begin{pmatrix} \lambda\cosh(P(t)) & \lambda\sinh(P(t)) & x(t) \\ \epsilon\lambda\sinh(P(t)) & \epsilon\lambda\cosh(P(t)) & y(t) \\ 0 & 0 & \epsilon \end{pmatrix}$$

$$(\lambda \;\; \text{constant})$$

while the triplet:

$$P^\alpha(t) = (\frac{\epsilon\dot{y}(t) - x(t)\dot{P}(t)}{2}, \frac{\epsilon\dot{x}(t) - y(t)\dot{P}(t)}{2}, -\frac{\epsilon\dot{P}(t)}{2})$$

where $\epsilon = \pm 1$.

For $h = \pm 1$, –class B–, the solutions are:

$$\Lambda^\alpha_\beta(t) = \begin{pmatrix} e^{-hP(t)}\lambda\cosh(P(t)) & e^{-hP(t)}\lambda\sinh(P(t)) & x(t) \\ e^{-hP(t)}\lambda\sinh(P(t)) & e^{-hP(t)}\lambda\cosh(P(t)) & c - hx(t) \\ 0 & 0 & 1 \end{pmatrix}$$

$$(\lambda \;\; \text{constant})$$

while the triplet:

$$\begin{aligned} P^\alpha(t) =& (\Omega(t), \frac{2h\Omega(t) - c\dot{P}(t) + 2hx(t)\dot{P}(t) + \dot{x}(t)}{2}, \\ & h\frac{e^{-hP(t)}\lambda\sinh(P(t))\dot{P}(t) - he^{-hP(t)}\lambda\cosh(P(t))\dot{P}(t)}{2e^{-hP(t)}\lambda\cosh(P(t)) - 2he^{-hP(t)}\lambda\sinh(P(t))}) \end{aligned}$$

For each of the previously mentioned cases, we have:

a) When $h = 0$, (class A), $\gamma_{\alpha\beta}$ can be made diagonal and then the shift vanishes. Thus the counting of the essential parameters is: 6 -2 (from the quadratic constraint) -1 (from the constant, contained in Λ^α_β) $= 3$.

b) When $h = \pm 1$, (class B), using $x(t)$ and $P(t)$, we can eliminate γ_{13} and γ_{23}. So: 8 -2 (from the quadratic constraint) -2 (from the constants, contained in Λ^α_β) $= 4$ is the number of the essential constants. Notice that the 3 linear constraint equations, are linearly dependent and thus, when $N^3 = 0$ through (2.14b), there is no linear constraint equation left, to remove any constants, hence the number 4.

c) When $h \neq 0, \pm 1$, the counting algorithm is exactly the same, as in Type IV case.

Type VII rank(m)=2, signature(m)=Euclidean and $\nu_\alpha \neq 0$. We may set $m^{\alpha\beta} = diag(-1, -1, 0)$, $\nu_\alpha = h\delta^3_\alpha$. For all values of h, equations (2.16), (2.17) give the following

form for $\Lambda^{\alpha}_{\beta}(t)$:

$$\Lambda^{\alpha}_{\beta}(t) = \begin{pmatrix} \lambda e^{hP(t)}\cos(P(t)) & \lambda e^{hP(t)}\sin(P(t)) & x(t) \\ -\lambda e^{hP(t)}\sin(P(t)) & \lambda e^{hP(t)}\cos(P(t)) & y(t) \\ 0 & 0 & 1 \end{pmatrix}$$

$$(\lambda \ \ \text{constant})$$

and the triplet:

$$\begin{aligned} P^{\alpha}(t) =& (\frac{x(t)\dot{P}(t) + h^2 x(t)\dot{P}(t) - h\dot{x}(t) + \dot{y}(t)}{2(1+h^2)}, \\ & \frac{y(t)\dot{P}(t) + h^2 y(t)\dot{P}(t) - h\dot{y}(t) - \dot{x}(t)}{2(1+h^2)}, \frac{\dot{P}(t)}{2}) \end{aligned}$$

For the case $h = 0$, there is another solution, except the one deduced from the previous, by setting $h = 0$:

$$\Lambda^{\alpha}_{\beta}(t) = \begin{pmatrix} \lambda\cos(P(t)) & \lambda\sin(P(t)) & x(t) \\ \lambda\sin(P(t)) & -\lambda\cos(P(t)) & y(t) \\ 0 & 0 & -1 \end{pmatrix}$$

$$(\lambda \ \ \text{constant})$$

and the triplet:

$$P^{\alpha}(t) = (\frac{x(t)\dot{P}(t) - \dot{y}(t)}{2}, \frac{y(t)\dot{P}(t) + \dot{x}(t)}{2}, -\frac{\dot{P}(t)}{2})$$

Again, for each of the previously mentioned cases, we have:

a) When $h = 0$, (class A), $\gamma_{\alpha\beta}$ can be made diagonal and equations (2.5b) give $N^a = 0$. Thus: $6 \ \ -2$ (from the quadratic constraint) $\ -1$ (from the constant, contained in Λ^{α}_{β})

$= 3$ is the number of the essential constants.

b) When $h \neq 0$, the counting algorithm is exactly the same, as in Type IV case.

For Bianchi Types VIII and IX, condition (2.17), does not impose any restriction on $\Lambda^{\alpha}_{\beta}(t)$, but rather fixes completely, the triplet $P^a(t)$, to be:

$$P^a = \frac{1}{4|m|}\varepsilon_{\beta\tau\kappa}m^{\alpha\beta}\Lambda^{\tau}_{\gamma}\dot{\Lambda}^{\kappa}_{\delta}m^{\gamma\delta}$$

Type VIII rank(m)=3, signature(m)=Lorentzian. A standard choice is $m^{\alpha\beta} = \eta^{\alpha\beta} = diag(-1,1,1)$. Since $|m^{\alpha\beta}| = -1$, (2.19a) implies that $|\Lambda^{\alpha}_{\beta}| = 1$ and thus, Λ^{α}_{β}'s are the isometries of the Minkowski metric, in three dimensions, i.e. the Lorentz boosts, with one timelike and two spacelike directions, times a rotation of the "space" plane. Thus, the automorphisms are characterized by the two components of the velocity vector, plus the rotation angle. The triplet P^a, is:

$$P^a = \frac{1}{2}(\Lambda^2_{\mu}\dot{\Lambda}^3_{\nu}\eta^{\mu\nu}, -\Lambda^3_{\mu}\dot{\Lambda}^1_{\nu}\eta^{\mu\nu}, -\Lambda^1_{\mu}\dot{\Lambda}^2_{\nu}\eta^{\mu\nu})$$

It can be proven –see appendix A– that a positive definite matrix, can be diagonalized by this automorphism group; i.e. we can set

$\gamma_{\alpha\beta} = diag(a^2(t), b^2(t), c^2(t))$. Then, from (2.5b), we will have $N^a = 0$.

The number 4, of the expected essential parameters –see table below–, can be understood as follows: The time-dependent Lorentz transformation Λ^{α}_{β}, can diagonalize $\gamma_{\alpha\beta}$. Thus, the counting: $6 - 2$ (from the quadratic constraint) $= 4$.

Type IX rank(m)=3, signature(m)=Euclidean. The standard choice is $m^{\alpha\beta} = \delta^{\alpha\beta}$. Since $|m^{\alpha\beta}| = 1$,(2.19a) implies that $|\Lambda^{\alpha}_{\beta}| = 1$ and thus, Λ^{α}_{β}'s are the isometries of the Eu-

clidean metric, in three dimensions, i.e. the orthogonal matrices, which are characterized by three parameters; e.g. the Euler angles. The triplet P^a, is:

$$P^a = \frac{1}{2}(\Lambda^2_\mu \dot{\Lambda}^3_\nu \delta^{\mu\nu}, \Lambda^3_\mu \dot{\Lambda}^1_\nu \delta^{\mu\nu}, \Lambda^1_\mu \dot{\Lambda}^2_\nu \delta^{\mu\nu})$$

Since a positive definite symmetric matrix can be diagonalized by an element of –the connected to the identity component of– $O(3)$, we have that $\gamma_{\alpha\beta}(t) = diag(a^2(t),\ b^2(t),\ c^2(t))$ [19]. Then, from (2.5b), as is well-known $N^\alpha = 0$.

The counting yields –exactly as in Type VIII: $6 - 2$ (from the quadratic constraint) $= 4$, essential constants.

¿From the above analysis of the space of solutions to (2.16) and (2.17), we observe that in each Bianchi Type, there are 3 arbitrary functions of time –as expected–, for a twofold reason;

Firstly, because we are solving the integrability conditions for the existence of a time-dependent spatial diffeomorphism according to (2.9).

Secondly, because as it has been mentioned in the Introduction, the system of Einstein's equations (2.5), has a gauge freedom of 4 arbitrary functions of time. But one of them, simply corresponds to time reparametrization, while the remaining 3, are the ones we found in the above analysis.

In the various Bianchi Types, the 3 arbitrary functions, are distributed differently among the components of $\Lambda^a_\beta(t)$ and $P^a(t)$. This fact, together with the different number of arbitrary constants appearing in Λ^a_β for each Type, results in a different number of essential constants –expected by independent arguments [19] to appear in the general

solutions to Einstein's equations (2.5) –see Table.

We now conclude section 2, by stating the following (uniqueness) Theorem:

"In a given –albeit arbitrary– Bianchi Type, let γ_1, γ_2, *(in matrix notation) be solutions to Einstein's equations (2.5), then there is a matrix* M *of the form:* $M = \Lambda_1^{-1}\Sigma\Lambda_2$ *(where* Λ_1 *and* Λ_2 *are solutions to (2.16) and (2.17) and* Σ, *represents the irrelevant symmetries of the solution in its irreducible form) which connects them as:* $\gamma_2 = M^T\gamma_1 M$.*"*

Note: N, N^a, are understood to be given from the quadratic and linear constraint equations (2.5a,b).

The proof rests on the previously established facts:

a) That the solutions to (2.16) and (2.17), suffice to reduce the generic $\gamma_{\alpha\beta}$, to a form that will contain the expected necessary number of essential constants, so as to be regarded as the most general one –for each and every Bianchi Type.

b) That for every given Bianchi Type, the solutions to (2.16) and (2.17), form a group.

Indeed, let γ_1, γ_2 be solutions to (2.5). Then there are Λ_1, Λ_2 –along with P_1, P_2 respectively, if needed– solutions to (2.16) and (2.17), such that:

$$\gamma_1 = \Lambda_1^T\gamma_{irreducible}\Lambda_1$$

$$\gamma_2 = \Lambda_2^T\gamma_{irreducible}\Lambda_2$$

where $\gamma_{irreducible}$, stands for the solution in a form exhibiting, only the essential constants.

From the first of these:

$$\gamma_{irreducible} = (\Lambda_1^{-1})^T\gamma_1\Lambda_1$$

Since, –by definition– $\gamma_{irreducible}$ is a symmetric matrix there are always, non-trivial matrices Σ, such that:

$$\gamma_{irreducible} = \Sigma^T \gamma_{irreducible} \Sigma$$

Substituting the two last in the second, we obtain:

$$\gamma_2 = (\Lambda_1^{-1} \Sigma \Lambda_2)^T \gamma_1 \Lambda_1^{-1} \Sigma \Lambda_2$$

q.e.d.

3 The Space of Solutions for Type II and V Cases

In this section, we adopt the more conventional point of view; that of "gauge" fixing, before solving. As far as time is concerned, we adopt the "gauge" fixing condition $\widetilde{N} = \sqrt{\widetilde{\gamma}}$, since this simplifies the form of the equations. For the spatial coordinates, as explained in the previous section, a choice of reference system, amounts to a choice of time-dependent automorphism –along with a choice of $P^\alpha(t)$–; thus, we select the form of $\widetilde{\gamma}_{\alpha\beta}(t)$ to be such that, the linear equation would imply $\widetilde{N}^\alpha = 0$. In this "gauge", Einstein's equations (2.5) read:

$$-\widetilde{\gamma}^{\alpha\kappa}\widetilde{\gamma}^{\beta\lambda}\dot{\widetilde{\gamma}}_{\kappa\lambda}\dot{\widetilde{\gamma}}_{\alpha\beta} + \left(\frac{\dot{\widetilde{\gamma}}}{\widetilde{\gamma}}\right)^2 - 4\widetilde{\gamma}R = 0 \tag{3.1a}$$

$$C^\epsilon_{\alpha\mu}\widetilde{\gamma}^{\mu\rho}\dot{\widetilde{\gamma}}_{\rho\epsilon} - C^\epsilon_{\mu\epsilon}\widetilde{\gamma}^{\mu\rho}\dot{\widetilde{\gamma}}_{\rho\alpha} = 0 \tag{3.1b}$$

$$\ddot{\widetilde{\gamma}}_{\alpha\beta} - \widetilde{\gamma}^{\mu\nu}\dot{\widetilde{\gamma}}_{\alpha\mu}\dot{\widetilde{\gamma}}_{\beta\nu} - 2\widetilde{\gamma}R_{\alpha\beta} = 0 \tag{3.1c}$$

Note that taking the trace of equations (3.1c), one arrives at:

$$(\frac{\dot{\widetilde{\gamma}}}{\widetilde{\gamma}})^{\cdot} - 2\widetilde{\gamma}R = 0 \tag{3.2}$$

A somewhat useful result deriving from (3.2), is the following: $\widetilde{\gamma} = ae^{\beta t}$ implies $\widetilde{R} = 0$, which is incompatible for all but I Bianchi Types.

We now present, a realization, of the method developed in the previous section, for the cases of Type II and V, Bianchi geometries. At this point, a word of caution is pertinent: it is evident –from the previously mentioned counting, of the expected number of essential constants–, that the well known Taub's (Type II) and Joseph's (Type V) solutions, are the most general for the respective cases [18]. Thus, we should not expect to find anything new –in this respect. However, the thorough investigation of the complete space of solutions, requires the knowledge of the correct (gauge) symmetry group for Einstein's equations (2.5). In this respect, we shall directly show, that transformations (2.14), –as specified by conditions (2.16) and (2.16), applied to Types II and V–, are essentialy, the only (gauge) symmetries of these Bianchi geometries.

Note: From now on we drop the tildes from the various quantities for simplicity –except in some cases, where misunderstanding could occur.

3.1 Bianchi Type II

As it can be seen, from the results concerning Type II, we can consider –without loss of generality–, the time-dependent part $\gamma_{\alpha\beta}(t)$, of the 3-metric, to have the form:

$$\gamma_{\alpha\beta}(t) = \begin{pmatrix} a(t) & 0 & 0 \\ 0 & b(t) & f(t) \\ 0 & f(t) & c(t) \end{pmatrix}$$

It is interesting to observe that, the freedom in arbitrary functions of time –contained in $\Lambda^{\alpha}_{\beta}(t)$–, does not suffice to diagonalize $\gamma_{\alpha\beta}(t)$, i.e. to set $f(t) = 0$, a priori. Yet, we know –see (3.16) and (3.17) below– that the diagonal Taub's metric, is the irreducible form of the most general Type II, solution. The reconciliation of these two, seemingly conflicting facts, obtains only on mass shell; after we have completely solved (3.1), with $\gamma_{\alpha\beta}(t)$ given above, $f(t)$ becomes linearly dependent upon $b(t)$ and $c(t)$, and we can thus, gauge it away –utilizing the remaining freedom in arbitrary constants, contained in $\Lambda^{\alpha}_{\beta}(t)$.

Note: From now on, we drop the t-symbol –for time dependence–, from the various quantities; e.g., a stands for $a(t)$.

Inserting the form of $\gamma_{\alpha\beta}$ in equations (3.1b), we find that they vanish identically. We next consider, the following quantity q, which is scalar under a general linear mixing $\sigma^{\alpha} \to \widetilde{\sigma}^{\alpha} = S^{\alpha}_{\beta}\sigma^{\alpha}$, with $S^{\alpha}_{\beta} \in GL(3, \Re)$,

$$q = C^{\kappa}_{\mu\nu} C^{\lambda}_{\tau\sigma} \gamma_{\kappa\lambda} \gamma^{\mu\tau} \gamma^{\nu\sigma} = \frac{a^2}{2\gamma} = \frac{a}{2(bc - f^2)}$$

where γ, as usual, denotes the determinant of the matrix $\gamma_{\alpha\beta}$. Then, (2.7)., gives the

following non-zero components for the Ricci tensor $R_{\alpha\beta}$, and the Ricci scalar, R:

$$\begin{aligned} R_{11} &= -q\gamma_{11} \\ R_{rs} &= q\gamma_{rs} \quad r,s = 2,3 \\ R &= q \end{aligned} \tag{3.3}$$

The (1,1) component of (3.1c), is an autonomous equation for the scale factor a:

$$(\frac{\dot{a}}{a})^{\cdot} + a^2 = 0 \tag{3.4}$$

with a first integral:

$$(\frac{\dot{a}}{a})^2 + a^2 = \omega = constant > 0 \tag{3.5}$$

Using (3.2), (3.3) and (3.4), we get the equation for q:

$$(\frac{\dot{q}}{q})^{\cdot} + 3a^2 = 0 \tag{3.6}$$

To obtain first integrals for (3.1c), let us define the new variables:

$$\begin{aligned} \overline{\gamma}_{11} = q^{-1/3}\gamma_{11} \quad \overline{\gamma}_{rs} = q^{1/3}\gamma_{rs} \\ \overline{\gamma}^{11} = q^{1/3}\gamma^{11} \quad \overline{\gamma}^{rs} = q^{-1/3}\gamma_{rs} \end{aligned} \tag{3.7}$$

Then:

$$\overline{\gamma} = det(\overline{\gamma}_{\alpha\beta}) = q^{1/3}\gamma = \frac{a^2}{2}q^{-2/3} \tag{3.8}$$

It is straightforward to see that, with the use of (3.7), and (3.3), (3.4), (3.6), the spatial Einstein's equations (3.1c), translate into the following simple, integrable, Kasner-like, equations, –in terms of $\overline{\gamma}_{\alpha\beta}$:

$$(\overline{\gamma}^{\alpha\rho}\dot{\overline{\gamma}}_{\rho\beta})^{\cdot} = 0 \tag{3.9}$$

with first integrals:

$$\overline{\gamma}^{\alpha\rho}\dot{\overline{\gamma}}_{\rho\beta} = \vartheta^{\alpha}_{\beta} \tag{3.10}$$

where:

$$\vartheta^{\alpha}_{\beta} = \begin{pmatrix} \theta^1_1 & 0 & 0 \\ 0 & \theta & \varrho \\ 0 & \sigma & \pi \end{pmatrix}$$

Taking the trace of (3.10), we obtain –by means of (3.8):

$$2(\frac{\dot{a}}{a} - \frac{1}{3}\frac{\dot{q}}{q}) = \vartheta^a_a = \theta^1_1 + \vartheta^s_s \tag{3.11}$$

while the (1,1) component of (3.10), gives:

$$\frac{\dot{a}}{a} - \frac{1}{3}\frac{\dot{q}}{q} = \theta^1_1 \tag{3.12}$$

The last two, imply that $\theta^1_1 = \vartheta^s_s = \theta + \pi$, so finally, the matrix ϑ becomes:

$$\vartheta^{\alpha}_{\beta} = \begin{pmatrix} \theta + \pi & 0 & 0 \\ 0 & \theta & \varrho \\ 0 & \sigma & \pi \end{pmatrix} \tag{3.13}$$

Using the relation $\gamma = a^2/(2q)$ –earlier mentioned– as well as (3.5) (3.6), (3.7) and (3.10), it is straightforward to see that the quadratic constraint equation (3.1a), becomes a relation among constants; that is:

$$\omega = 2(\vartheta^s_s)^2 + |\vartheta^r_s| \tag{3.14}$$

Integrating (3.5), we get the scale factor a:

$$a(t) \doteq a = \frac{\sqrt{\omega}}{\cosh(\pm\sqrt{\omega}t)} \tag{3.15}$$

¿From this relation and (3.12), we conclude that:

$$q^{-1/3}a = a_0 e^{\vartheta^s_s t}, \quad a_0 > 0 \tag{3.16}$$

Now utilizing, in matrix notation, the relation: $\overline{\gamma}\vartheta = \vartheta^T\overline{\gamma}$ –which is the consistency requirement for (3.10)– and (3.16), we deduce that classical solutions exist, only for matrices ϑ, with real eigenvalues –and thus diagonalizable. Since (2.16), (2.17) admit the solutions Λ^α_β=constant, $P^a = 0$, we can invoke a constant mixing of ϑ, with a matrix of the form:

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Lambda^2_2 & \Lambda^2_3 \\ 0 & \Lambda^3_2 & \Lambda^3_3 \end{pmatrix}$$

and reduce it, to a diagonal form. Then, we are essentially led to the diagonal Taub's solution:

$$\begin{aligned} \overline{\gamma}_{22} &= q^{1/3}b = e^{\theta t} \\ \overline{\gamma}_{33} &= q^{1/3}c = e^{\pi t} \end{aligned} \tag{3.17}$$

At this point, it is interesting to observe that, if for some reason, we had not invoked this diagonalizing Λ, and instead proceeded with the general ϑ^r_s, we would had arrived, at a reducible form of the solutions with a non-vanishing $\overline{\gamma}_{23}$. However, this off-diagonal element, can be made to vanish through the action of the, previously mentioned, Λ.

Thus, we have shown that "gauge" transformations (2.14) –with (2.16) and (2.17), holding–, suffice to reduce the most general line element, for the Type II Bianchi Model, to the known Taub's metric. According to the theorem stated at the end of section 2, these transformations are, essentially, unique. We are now going to explicitly verify it –for the case at hand.

A convenient way to proceed, is to start from Taub's form of the solution and ask ourselves, what is the form of the most general time-dependent automorphism $\Lambda^{\alpha}_{\beta}(t)$, which retains the form invariance of Einstein's equations (2.5) –written in the invariant basis. Since we know that Λ^1_2 and Λ^1_3, can be time-dependent, we focus on a time-dependent matrix Λ, of the form:

$$\Lambda^{\alpha}_{\beta} = \begin{pmatrix} \varrho & 0 & 0 \\ 0 & \varrho_1 & \varrho_2 \\ 0 & \varrho_3 & \varrho_4 \end{pmatrix} \tag{3.18}$$

where: $\varrho = \varrho_1\varrho_4 - \varrho_2\varrho_3$, and all ϱ's, are time-dependent.

Consider the transformation, induced by this Λ^{α}_{β}, on $\gamma^{Taub}_{\alpha\beta}$ –in matrix notation:

$$\widehat{\gamma} = \Lambda^T \gamma^{Taub} \Lambda \tag{3.19}$$

The linear constraint equations (3.1b), still imply $\widehat{N}^a = 0$. As far as the time gauge fixing condition $N = \sqrt{\gamma}$ is concerned, we have: $\sqrt{\widehat{\gamma}} = |\Lambda|\sqrt{\gamma^{Taub}}$, $|\Lambda| > 0$, and thus:

$$N^{Taub} dt^{Taub} = \widehat{N} d\widehat{t} \Rightarrow$$

$$d\widehat{t}|\Lambda|\sqrt{\gamma^{Taub}} = \sqrt{\gamma^{Taub}} dt^{Taub} \Rightarrow$$

$$d\widehat{t}\varrho^2 = dt^{Taub}$$

Since we wish for the transformation, to be a symmetry of (2.5), and we have secured that $\widehat{N}^a = 0$, and selected $\widehat{N} = \sqrt{\widehat{\gamma}}$, the equation satisfied by $\widehat{\gamma}_{\alpha\beta}$, would be exactly (3.1) and (3.2). Only the dot –defining the time derivative, with respect to Taub's time–, must be replaced by a prime:

$$ ' \doteq \frac{d}{d\widehat{t}} = \varrho^2(t^{Taub}) \frac{d}{dt^{Taub}} = \varrho^2(t^{Taub}) \times^{\cdot} \tag{3.20} $$

Defining the corresponding scale quantities $\widehat{\widehat{\gamma}}_{\alpha\beta}$, –according to (3.7) and (3.8)– we must have the analogues of (3.10):

$$ \widehat{\widehat{\gamma}}^{\alpha\rho} \widehat{\widehat{\gamma}}\,'_{\rho\beta} = \vartheta^{\alpha}_{\beta} \tag{3.21} $$

Equation (3.2), reads:

$$ (\frac{\widehat{\gamma}\,'}{\widehat{\gamma}})\,' - 2\widehat{\gamma}\widehat{R} = 0 \tag{3.22} $$

It also holds:

$$ (\frac{\dot{\gamma}_{Taub}}{\gamma_{Taub}})^{\cdot} - 2\gamma_{Taub} R_{Taub} = 0 \tag{3.23} $$

Translating (3.22) in the t^{Taub}-variable, and subtracting (3.23), we get:

$$ 2(\varrho^2)^{\cdot\cdot} + (\varrho^2)^{\cdot} \frac{\dot{\gamma}_{Taub}}{\gamma_{Taub}} = 0 $$

which, with the help of $\gamma_{Taub} = a^2_{Taub}/2q$, (3.12) and $(\theta^1_1)^{Taub} = (\vartheta^s_s)^{Taub}$, becomes:

$$ 2(\varrho^2)^{\cdot\cdot} + (\varrho^2)^{\cdot}(2(\vartheta^s_s)^{Taub} - \frac{1}{3}\frac{\dot{q}}{q}) = 0 \tag{3.24} $$

The (1,1) component of (3.21) is:

$$ \frac{\widehat{\widehat{\gamma}}_{11}\,'}{\widehat{\widehat{\gamma}}_{11}} = \vartheta^s_s $$

where:

$$\widehat{\widehat{\gamma}}_{11} = q^{-1/3}\widehat{\gamma}_{11} = q^{-1/3}\varrho^2\gamma_{11}^{Taub}$$

and thus, that component reads:

$$(\varrho^2)^{\cdot} + \varrho^2(\vartheta^s_s)^{Taub} = \vartheta^s_s \tag{3.25}$$

Inserting the derivative of (3.25) into (3.24), we have:

$$(\varrho^2)^{\cdot}\frac{\dot{q}}{q} = 0$$

which in conjunction with (3.6), implies ϱ^2=constant. Without loss of generality, we can take ϱ^2=1. Henceforth, the time variable $\widehat{t}$, may –and will– denote Taub's time.

It is thus left for us, to investigate the unimodular matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \varrho_1 & \varrho_2 \\ 0 & \varrho_3 & \varrho_4 \end{pmatrix}$$

with: $1 = \varrho_1\varrho_4 - \varrho_2\varrho_3$, and all ϱ's, are time-dependent.

It can be proved that a convenient parametrization for this task, is:

$$\Lambda^\alpha_\beta = \begin{pmatrix} 1 & 0 \quad 0 \\ 0 & \\ & \Lambda^r_s \\ 0 & \end{pmatrix}$$

where:

$$\Lambda^r_s = R^r_m L^m_s$$

$$L_s^m = \begin{pmatrix} \varphi(t) & \chi(t) \\ 0 & 1/\varphi(t) \end{pmatrix}$$

and R_m^r, are the symmetries of the Taub's metric, i.e.

$R^T \gamma^{Taub} R = \gamma^{Taub}$ –in matrix notation–:

$$R = \begin{pmatrix} 1 & 0 \quad 0 \\ 0 & \\ & R_m^r \\ 0 & \end{pmatrix}$$

R_m^r being:

$$R_m^r = \begin{pmatrix} \cos(\tilde{g}(t)) & \sin(\tilde{g}(t)) e^{-(\kappa-\mu)t/2} \\ -\sin(\tilde{g}(t)) e^{(\kappa-\mu)t/2} & \cos(\tilde{g}(t)) \end{pmatrix}$$

where $\tilde{g}(t)$, is an unspecified function of time, and $\kappa,\ \mu$, the eigenvalues of ϑ^{Taub}.

The system (3.21), gives the following equations for $\chi(t)$ and $\varphi(t)$:

$$2\frac{\dot{\varphi}}{\varphi} + \kappa = \theta + \sigma\frac{\chi}{\varphi} \tag{3.26a}$$

$$(\frac{\chi}{\varphi})^{\cdot} = -\sigma(\frac{\chi}{\varphi})^2 + (\pi - \theta)\frac{\chi}{\varphi} + \varrho \tag{3.26b}$$

$$e^{(\kappa-\mu)t}(\dot{\chi}\varphi - \chi\dot{\varphi}) = \frac{\sigma}{\varphi^2} \tag{3.26c}$$

$$-2\frac{\dot{\varphi}}{\varphi} + \mu = \pi - \sigma\frac{\chi}{\varphi} \tag{3.26d}$$

Equation (3.25), for the choice ϱ^2=1, gives $(\vartheta_s^s)^{Taub} = \vartheta_s^s$, and hence:

$$\pi + \theta = \kappa + \mu \tag{3.27}$$

It also implies, $\widehat{\widehat{\gamma}}_{11} = \overline{\gamma}_{11}^{Taub}$, or $a(t) = a^{Taub}(t)$, as well as, $\omega = \omega^{Taub}$, or –through (3.14)–:

$$2(\theta+\pi)^2 + \pi\theta - \varrho\sigma = 2(\kappa+\mu)^2 + \kappa\mu \xrightarrow{3.27} \kappa\mu = \theta\pi - \varrho\sigma \tag{3.28}$$

Out of the 4 differential equations (3.26), only the first three, are independent –in view of (3.27). The solution to this system, for $\sigma \neq 0$, is given by:

$$\frac{\chi}{\varphi} = k_1 - \frac{\lambda^3 c^4 e^{-\lambda t}}{\sigma(1+\lambda^2 c^4 e^{-\lambda t})} \quad \lambda = \kappa - \mu \tag{3.29}$$

–from the Riccati (3.26b), where $k_1 = (\pi - \theta + \lambda)/2\sigma$, is the constant special solution and:

$$\varphi^2 = \frac{\sigma}{\lambda^2 c^2}(1+\lambda^2 c^4 e^{-\lambda t}) \quad \sigma > 0 \tag{3.30}$$

Thus, it is easily seen that, (3.29) and (3.30) make the matrix L^m_s, to be written in the form $L^m_s = \Sigma^m_n \widetilde{L}^n_s$, where:

$$\Sigma^m_n = \begin{pmatrix} \cos(g(t)) & \sin(g(t))e^{-\lambda t/2} \\ \sin(g(t))e^{\lambda t/2} & -\cos(g(t)) \end{pmatrix}$$

$$\widetilde{L}^n_s = \begin{pmatrix} \varepsilon_1 \frac{\sqrt{\sigma}}{\lambda c} & k_1 \varepsilon_1 \frac{\sqrt{\sigma}}{\lambda c} \\ \varepsilon_2 c\sqrt{\sigma} & (k_1 - \frac{\lambda}{\sigma})\varepsilon_2 c\sqrt{\sigma} \end{pmatrix}$$

with $(\varepsilon_1)^2 = (\varepsilon_2)^2 = 1$, $(\sigma,\ c) > 0$ and:

$$\tan(g(t)) = \frac{\varepsilon_2 c^2 \lambda}{\varepsilon_1} e^{-\lambda t/2}$$

There are the special cases $\sigma = 0$, or $\lambda = 0$, which are easily seen, to fall into the previous case.

Thus, in all cases, there always exist matrices Σ and $\widetilde{L}$, such that the transformation matrix Λ^{α}_{β}, can be written as:

$$\Lambda^{\alpha}_{\beta} = \begin{pmatrix} 1 & 0 \quad 0 \\ 0 & \\ & R^r_m \Sigma^m_n \widetilde{L}^n_s \\ 0 & \end{pmatrix}$$

This concludes the verification of the Theorem stated at the end of the section 2, since indeed, R and Σ, have trivial action, on $\gamma^{Taub}_{\alpha\beta}$. It is therefore, evident that the most general $\gamma_{\alpha\beta}$, $N(t)$ and $N^a(t)$, satisfying equations (2.5), are –in matrix notation:

$$\gamma_{most\ general}(t) = \Lambda^T(t)\gamma^{Taub}(h(t))\Lambda(t)$$

$$\Lambda = \begin{pmatrix} \varrho_1\varrho_4 - \varrho_2\varrho_3 & x(t) & y(t) \\ 0 & \varrho_1 & \varrho_2 \\ 0 & \varrho_3 & \varrho_4 \end{pmatrix}$$

where, the ϱ's are constant, and:

$$N(t) = \sqrt{|\gamma_{most\ general}|}\dot{h}(t)$$

$$N^{\alpha}(t) = S^{\alpha}_{\beta}(t)P^{\beta}(h(t))\dot{h}(t)$$

$$P^{\beta}(h(t)) = \{P(t), \frac{\varrho_1\dot{y}(t) - \varrho_2\dot{x}(t)}{(\varrho_1\varrho_4 - \varrho_2\varrho_3)\dot{h}(t)}, \frac{\varrho_3\dot{y}(t) - \varrho_4\dot{x}(t)}{(\varrho_1\varrho_4 - \varrho_2\varrho_3)\dot{h}(t)}\}$$

$$S = \Lambda^{-1}(t)$$

$$\gamma^{Taub}(h(t)) = \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{e^{(2\kappa+\mu)h(t)}}{a} & 0 \\ 0 & 0 & \frac{e^{(\kappa+2\mu)h(t)}}{a} \end{pmatrix}$$

$$a = \frac{\sqrt{\omega}}{\cosh(\pm\sqrt{\omega}h(t))}$$

$$\omega = 2(\kappa+\mu)^2 + \kappa\mu$$

where the fourth arbitrary function $h(t)$, accounts for the time reparametrization covariance, i.e. permits us to depart from the time gauge fixing $N = \sqrt{\gamma}$.

3.2 Bianchi Type V

As it can be seen, from the results of section 2, concerning Type V, we can consider –with the usage of time-dependent A.I.D.'s–, the time-dependent part $\gamma_{\alpha\beta}(t)$, of the 3-metric, to be of the form:

$$\gamma_{\alpha\beta}(t) = \begin{pmatrix} a(t) & b(t) & 0 \\ b(t) & c(t) & 0 \\ 0 & 0 & f(t) \end{pmatrix}$$

with $a(t)c(t) - b^2(t) = f^2(t)$. Again, as it happens for Type II, the form of the allowed transformation $\Lambda^{\alpha}_{\beta}(t)$ is such that, one can not set $b(t) = 0$, a priori. Yet, we know –see (3.39) and (3.40) below– that the diagonal Joseph's metric, is the irreducible form of the most general Type V, solution. This puzzle, finds its resolution only on mass shell; after we have completely solved (3.1) with $\gamma_{\alpha\beta}(t)$ given above, $b(t)$ becomes linearly dependent

upon $a(t)$ and $c(t)$, and we can thus, put it to zero –utilizing the remaining freedom in arbitrary constants, contained in $\Lambda^{\alpha}_{\beta}(t)$.

Note: From now on, we drop the t-symbol –for time dependence–, from the various quantities; e.g., a stands for $a(t)$.

Inserting the form of $\gamma_{\alpha\beta}$ in equations (3.1b), we find that they vanish identically. We next define, the scalar –under a general linear mixing $\sigma^{\alpha} \to \widetilde{\sigma}^{\alpha} = S^{\alpha}_{\beta}\sigma^{\alpha}$, with $S^{\alpha}_{\beta} \in GL(3, \Re)$– quantity q:

$$q = C^{\tau}_{\tau\mu} C^{\sigma}_{\sigma\nu} \gamma^{\mu\nu} = \frac{1}{f}$$

The condition $ac - b^2 = f^2$, now reads as: $ac - b^2 = 1/q^2$, or

$$\gamma = \frac{1}{q^3} \tag{3.31}$$

Then, (2.7), gives:

$$\begin{aligned} R_{\alpha\beta} &= 2q\gamma_{\alpha\beta} \\ R &= 6q \end{aligned} \tag{3.32}$$

The (3,3) component of (3.1c), gives an autonomous equation for the scalar quantity q:

$$(\frac{\dot{q}}{q})^{\cdot} + \frac{4}{q^2} = 0 \tag{3.33}$$

with a first integral:

$$(\frac{\dot{q}}{q})^2 - \frac{4}{q^2} = \omega = constant \tag{3.34}$$

Defining the scaled quantities:

$$\begin{aligned} \overline{\gamma}_{\alpha\beta} &= q\gamma_{\alpha\beta} \\ \overline{\gamma}^{\alpha\beta} &= \tfrac{1}{q}\gamma^{\alpha\beta} \\ |\overline{\gamma}| &= 1 \end{aligned} \tag{3.35}$$

and using (3.32), (3.33), equations (3.1c), are translated into the following form:

$$(\overline{\gamma}^{\alpha\rho}\dot{\overline{\gamma}}_{\rho\beta})^{\cdot} = 0 \tag{3.36}$$

with first integrals:

$$\overline{\gamma}^{\alpha\rho}\dot{\overline{\gamma}}_{\rho\beta} = \vartheta^{\alpha}_{\beta} \tag{3.37}$$

where:

$$\vartheta^{\alpha}_{\beta} = \begin{pmatrix} \theta & \varrho & 0 \\ \sigma & -\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The form of the matrix ϑ, has been derived, using the form of $\overline{\gamma}_{\alpha\beta}$ and the property that $|\overline{\gamma}| = 1$. Using (3.31), (3.34) and (3.37), the quadratic constraint (3.1a), becomes a relation, among constants –as it was expected–, namely:

$$3\omega = \theta^2 + \varrho\sigma \tag{3.38}$$

The property $|\overline{\gamma}| = 1$, together with the consistency requirement –in matrix notation– $\overline{\gamma}\vartheta = \vartheta^T\overline{\gamma}$, which follows from (3.37), enables us to conclude that classical solutions, exist only for those values of the parameters, θ, ϱ, σ, for which ϑ, is diagonalizable, i.e. when $\theta^2 + \varrho\sigma > 0$.

Since the matrices Λ^{α}_{β}, of the form:

$$\Lambda^{\alpha}_{\beta} = \begin{pmatrix} \varrho_1 & \varrho_2 & 0 \\ \varrho_3 & \varrho_4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

along with $P^a = 0$, constitute the remaining gauge freedom, we can invoke such Λ^{α}_{β}, to diagonalize $\vartheta^{\alpha}_{\beta}$, and –at the same time– retain the shift, zero –see (2.14). Now with a diagonal $\vartheta^{\alpha}_{\beta}$, equations (3.37), essentially imply that $\overline{\gamma}_{\alpha\beta}$, is diagonal too.

A further integration of (3.34), yields:

$$\frac{1}{f(t)} = q(t) = \begin{cases} \frac{2}{\sqrt{\omega}} \sinh(\pm\sqrt{\omega}t) & \omega > 0 \\ \pm 2t & \omega = 0 \end{cases} \tag{3.39}$$

and thus, we are laid to the well known Joseph's solution –through complete integration of (3.37), for the diagonal case:

$$\begin{aligned} \overline{\gamma}_{11} &= qa = e^{\lambda t} \\ \overline{\gamma}_{22} &= qc = e^{-\lambda t} \\ 3\omega &= \lambda^2 > 0 \end{aligned} \tag{3.40}$$

or the Milnor's solution [18], when $\omega = 0$ –with the corresponding q.

Once again, it is interesting to observe that if, for some reason, we do not invoke this diagonalizing Λ^{α}_{β} and, instead, proceed with the general $\vartheta^{\alpha}_{\beta}$, we arrive at a reducible form of the solution, which contains a non-vanishing $\overline{\gamma}_{12}$. However, this off-diagonal element, can be made to vanish through the action of the –previously mentioned– constant automorphism.

Thus, we have shown, that the "gauge" transformations (2.14), –with (2.16) and (2.17), holding– suffice to reduce the most general line element for the Type V Bianchi Model, to the known Joseph's metric, as predicted from the theorem, stated at the end of section 2. As we have done for the Type II case, we are now going to explicitly verify that these transformations, are essentially unique. To this end, let us consider the most general time-dependent automorphism, complementary to the time-dependent automorphism, described in section 2 –for the Type V, case.

$$\Lambda^\alpha_\beta = \begin{pmatrix} A(t) & B(t) & 0 \\ C(t) & F(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.41}$$

with $A(t)F(t)-B(t)C(t) = 1$. The action of such automorphism on $\gamma^{Joseph}_{\alpha\beta}$, is –in matrix notation:

$$\widehat{\gamma} = \Lambda^T \gamma^{Joseph} \Lambda$$

If we insert $\widehat{\gamma}_{\alpha\beta}$, in the linear constraint equations (3.1b), we learn that $\widehat{N}^a$, are also zero and, since, $|\widehat{\gamma}_{\alpha\beta}| = |\Lambda|^2 |\gamma^{Joseph}_{\alpha\beta}| = |\gamma^{Joseph}_{\alpha\beta}|$, we conclude that we are in the same temporal, as well as spatial, gauge. Therefore, $\widehat{\gamma}_{\alpha\beta}$, will also satisfy equations (3.1c). Since Λ^α_β, is an automorphism, it is a symmetry of q and thus, if we define the scaled quantities:

$$\widehat{\overline{\gamma}}_{\alpha\beta} = q\widehat{\gamma}_{\alpha\beta}$$

they must satisfy, the relations analogous to (3.37):

$$\widehat{\overline{\gamma}}^{\alpha\rho} \dot{\widehat{\overline{\gamma}}}_{\rho\beta} = \vartheta^\alpha_\beta \tag{3.42}$$

where:

$$\vartheta^{\alpha}_{\beta} = \begin{pmatrix} \theta & \varrho & 0 \\ \sigma & -\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

while, $\overline{\gamma}^{Joseph}_{\alpha\beta}$, satisfies the relations:

$$(\overline{\gamma}^{\alpha\rho})_{Joseph}(\dot{\overline{\gamma}}_{\rho\beta})_{Joseph} = (\vartheta^{\alpha}_{\beta})_{Joseph}$$

where:

$$(\vartheta^{\alpha}_{\beta})_{Joseph} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

By virtue of (3.34), –and since q, is invariant–, we get that $\omega = \omega^{Joseph}$, i.e.

$$\theta^2 + \varrho\sigma = \lambda^2 \tag{3.43}$$

In order to proceed with the integration of (3.42), it is convenient, to parametrize Λ^{α}_{β} in (3.41), as follows:

$$\Lambda^{\alpha}_{\beta} = \begin{pmatrix} \begin{matrix} & \\ & \Lambda^r_s & \\ & \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}$$

with $\Lambda^r_s = R^r_m L^m_s$, where R^r_m is:

$$\begin{pmatrix} e^{-\lambda t/2} & 0 \\ 0 & e^{\lambda t/2} \end{pmatrix} \cdot \begin{pmatrix} \cos(g(t)) & \sin(g(t)) \\ -\sin(g(t)) & \cos(g(t)) \end{pmatrix} \cdot \begin{pmatrix} e^{\lambda t/2} & 0 \\ 0 & e^{-\lambda t/2} \end{pmatrix}$$

i.e. the symmetries of the Joseph's metric; –in matrix notation

$R^T \gamma^{Joseph} R = \gamma^{Joseph}$ and L_s^m is:

$$\begin{pmatrix} \varphi(t) & \tau(t) \\ 0 & 1/\varphi(t) \end{pmatrix}$$

The system (3.42), gives the following differential equations for $\varphi(t)$ and $\tau(t)$:

$$2\frac{\dot{\varphi}}{\varphi} + \lambda = \theta + \sigma\frac{\tau}{\varphi} \tag{3.44a}$$

$$\dot{\tau}\varphi - \tau\dot{\varphi} = \varrho\varphi^2 - 2\theta\varphi\tau - \sigma\tau^2 \tag{3.44b}$$

$$e^{2\lambda t}(\dot{\tau}\varphi - \tau\dot{\varphi}) = \frac{\sigma}{\varphi^2} \tag{3.44c}$$

The solution to this system, for $\sigma \neq 0$, leads to incompatibility of the form $\varphi^2 = -e^2$, e a function of time.

For $\sigma = 0$, we get:

$$\begin{aligned} \varphi(t) &= c_1 e^{\frac{\theta-\lambda}{2}t} \\ \tau(t) &= c_1 \tfrac{\varrho}{2\theta} e^{\frac{\theta-\lambda}{2}t} \end{aligned} \tag{3.45}$$

with $c_1 > 0$, and –from (3.43), for the case at hand–, $\theta = \pm\lambda$. The case $\theta = \lambda$, trivially gives, a constant matrix

$$L_s^m = \begin{pmatrix} c_1 & c_1 \frac{\varrho}{2\lambda} \\ 0 & 1/c_1 \end{pmatrix}$$

while, the case $\theta = -\lambda$, gives

$$L_s^m = \begin{pmatrix} 0 & e^{-\lambda t} \\ e^{\lambda t} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1/c_1 \\ c_1 & -c_1 \frac{\varrho}{2\lambda} \end{pmatrix}$$

Since the first matrix in the product, is a symmetry of $(\overline{\gamma}_{\alpha\beta})_{Joseph}$, we again conclude that, the non-trivial action of Λ^{α}_{β}, on $(\overline{\gamma}_{\alpha\beta})_{Joseph}$, is tantamount to the action of a constant matrix in accordance to the theorem of section 2.

Finally, the most general line element $(\gamma_{\alpha\beta},\ N,\ N^{a})$ satisfying Einstein's equations (2.5), is thus given, –in matrix notation by:

$$\gamma_{most\ general}(t) = \Lambda^{T}(t)\gamma^{Joseph}(h(t))\Lambda(t)$$

$$\Lambda = \begin{pmatrix} \varrho_1 P(t) & \varrho_2 P(t) & x(t) \\ \varrho_3 P(t) & \varrho_4 P(t) & y(t) \\ 0 & 0 & 1 \end{pmatrix}$$

where the ϱ's are constant, subject to the condition $\varrho_1\varrho_4 - \varrho_2\varrho_3 = 1$ and:

$$N(t) = \sqrt{|\gamma_{most\ general}|}\dot{h}(t)$$

$$N^{\alpha}(t) = S^{\alpha}_{\beta} P^{\beta}(h(t))$$

$$P^{\beta}(h(t)) = \{x(t)(\ln\frac{x(t)}{P(t)})^{\cdot}, y(t)(\ln\frac{y(t)}{P(t)})^{\cdot}, (\ln\frac{1}{P(t)})^{\cdot}\}$$

$$S = \Lambda^{-1}$$

$$\gamma^{Joseph}(h(t)) = \begin{pmatrix} \frac{e^{\lambda h(t)}}{q} & 0 & 0 \\ 0 & \frac{e^{-\lambda h(t)}}{q} & 0 \\ 0 & 0 & f \end{pmatrix}$$

$$\frac{1}{f(h(t))} = q(h(t)) = \begin{cases} \frac{2}{\sqrt{\omega}}\sinh(\pm\sqrt{\omega}h(t)) & \omega > 0 \\ \pm 2h(t) & \omega = 0 \end{cases} \tag{3.46}$$

$$3\omega = \lambda^2$$

where the fourth arbitrary function $h(t)$, accounts for the time reparametrization covariance.

4 Discussion

In this work, we present an approach to the problem of solving Einstein's equations, for the case of a generic Bianchi-Type spatially homogeneous spacetime. The approach is not plagued by the fragmentation characterizing the major part of the existing rich literature –which is inherited by the diversity of the various simplifying ansatzen, employed in each case. The key notion for avoiding this fragmentation, is that of a Time-Dependent Automorphism Inducing Diffeomorphism; that is, a general coordinate transformation (2.9), mixing space and time coordinates, whose action on the line-element of a Bianchi Geometry, is described by relations (2.14) –viewed as "gauge" transformation laws for the dependent variables $\gamma_{\alpha\beta}(t)$, $N(t)$ and $N^{\alpha}(t)$. The investigation for the existence of such G.C.T.'s, leads to the necessary and sufficient conditions (2.16), (2.17); hence the name Time-Dependent A.I.D.'s. In each and every Bianchi Type, these conditions possess a non-empty set of solutions containing precisely three arbitrary functions of time. A choice of these arbitrary functions, amounts exactly to a choice for the three

spatial coordinates. Thus, the possibility is offered for simplifying Einstein equations, –through a simplification of $\gamma_{\alpha\beta}, N^{\alpha}, N$–, without running the risk of loss of generality or any sort of incompatibility.

Of course, the possible simplifications differ from one Bianchi Type to another; even within the same Bianchi Type, there are many possible simplifications –since one, can use the three arbitrary functions at will. This kinematical freedom, when combined to the dynamical information –furnished by the linear constraint equations–, considerably simplifies the form of the line-element and thus of Einstein's equations, as well. A useful, in our opinion, irreducible form of the line-element for each Bianchi Type, is given at the balance of section 2.

A statement that applies to all Types is that, using two of the three arbitrary functions, the scale-factor matrix $\gamma_{\alpha\beta}(t)$ can always –a priori; i.e. before solving any classical equations of motion– be put into a so called "symmetric" [16] form, i.e. $\gamma_{13} = \gamma_{23} = 0$. This applies also for Type II, if we take instead of the standard form for the structure constants ($C^1_{23} = -C^1_{32} = 1$, all other vanish) the equivalent version $C^3_{12} = -C^3_{21} = 1$, all other vanish. If this "symmetric" form, is then substituted into the linear equations, and the third arbitrariness is used, considerable restrictions among N^{α}'s and the remaining $\gamma_{\alpha\beta}$'s are obtained, as presented in detail at the end of section 2. Furthermore, with the help of the essential arbitrary constants in Λ^{α}_{β}, we can diagonalize $\gamma_{\alpha\beta}(t)$, on mass-shell. For all Bianchi Types, the shift vector N^{α}, can always be set to zero –with the help of Time-Dependent A.I.D.'s, and the linear equations. One could of course, rely on the well-known existence of Gauss-normal coordinates [2], and argue that this should be

true. However, in this work, the explicit realization of this fact is presented; what is more important, is that the vanishing of N^{α}, is accomplished without spoiling manifest spatial homogeneity. The interplay between line-elements with and without shift, established through Time-Dependent A.I.D.'s –see (2.14b)–, raises the need to reexamine the set of existing solutions -with respect to physical equivalence, among each other. In particular, many tilded and untilded fluid solutions [18], may proven to be G.C.T. related –and thus physically indistinguishable.

Except of the three arbitrary functions of time, of considerable importance, are also the (non absorbable in the shift) arbitrary constants, appearing in the solutions to (2.16) and (2.17). The number of these constants, varies for different Bianchi Types. The very interesting fact, is that when this number is subtracted from the number of constants, given by Peano's theorem, –after the freedom in arbitrary functions of time, has been fully exhausted–, the resulting number of the –finally– remaining constants, equals, for each and every Bianchi Type, to the number of expected essential constants –see [19], p. 211. This, permits us to conclude that the gauge symmetry transformations (2.14) –with (2.16) and (2.17), holding– are, essentially, unique. It is also noteworthing, that the existence of these constant parameters, helps to rectify a defect from which, the previous approach of Jantzen, is suffering; that of an uneven passage, from the lower to the higher Bianchi Types, owing to the change of the dimension of the invoked symmetry group [19]; indeed, the arbitrary functions of time are thus varying with dim[SAut(G)], from 8 (Type I), to 5 (Type II and V), to 3 (higher Types). This situation, is rather unsatisfactory, since we know that the independent or dynamical degrees of freedom

for the gravitational field, are 2 –per space point. Thus, in cosmology, we expect 2 independent functions of time –irrespective of Bianchi Type.

In contrast to this state of affairs, the solutions to (2.16) and (2.17), contain exactly 3 arbitrary functions of time, which together with the arbitrary function –owing to the time reparametrization covariance of equations (2.5)–, leave us with $6(\gamma_{\alpha\beta}) - 4 = 2$ arbitrary functions, in all Bianchi Types. The required sensitivity, of the method, to the particular isometry group, is represented by the extra constant parameters –as explained.

It is in this remarkable way, that General Relativity manages to encode the memory of spatial G.C.T. covariance, in the set of the reduced equations (2.5), where only functions of time and their derivatives appear. This encoding also persists in the actions –when these actions exist–, and leads to important grouping of $\gamma_{\alpha\beta}$'s, into the three solutions: $x^1 = C^\alpha_{\mu\nu} C^\beta_{\rho\sigma} \gamma^{\mu\rho} \gamma^{\nu\sigma} \gamma_{\alpha\beta}$, $x^2 = C^\alpha_{\beta\delta} C^\delta_{\nu\alpha} \gamma^{\beta\nu}$, $x^3 = \gamma$ of the quantum linear constraints [13, 20]. When a truly scalar Hamiltonian exists [13, 21], the wavefunction depends only on the q^i's:

$$q^1 = \frac{m^{\alpha\beta}\gamma_{\alpha\beta}}{\sqrt{\gamma}}, \quad q^2 = \frac{(m^{\alpha\beta}\gamma_{\alpha\beta})^2}{2\gamma} - \frac{1}{4} C^\alpha_{\mu\nu} C^\beta_{\rho\sigma} \gamma^{\mu\rho} \gamma^{\nu\sigma} \gamma_{\alpha\beta}, \quad q^3 = \frac{m}{\sqrt{\gamma}}$$

which completely and irreducibly, determine a spatial three-geometry.

To summarize, the system (2.5), admits solutions containing in each and every Bianchi Type, exactly four unspecified functions of time. One, corresponds to the freedom of changing the time coordinate; three, correspond to the freedom of changing the spatial coordinates via Time-Dependent A.I.D.'s. The action of such a transformation on the line-element, and on the system of equations (2.5), is described by relations (2.14),

(2.18). Thus, one does not actually need to calculate the simplifying G.C.T.'s; one simply uses (2.14), simplifies the equations, solves them completely, and finally inverts the transformation thereby obtaining the entire space of solutions. It is in this sense, that the closed form of the line elements presented in section 3, exhaust the space of classical solutions –for the case of Bianchi types II and V.

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The number of arbitrary constants appearing in general solution for each Bianchi Type –vacuum model–, is given in the following table –depicted in the first of [19], pp. 211:

Bianchi Type	# of the essential constants
I	1
II	2
VI_0, VII_0	3
VIII, IX	4
IV	3
V	1
VI_h ($h \neq$ -1/9)	3
$VI_{-1/9}$	4
VII_h	3

A Appendix

In [22], the following Theorem, is given:

"Let two symmetric forms A and B, be given, on a n-dimensional linear vector space V. If one of them –say A– is non singular, then there is a base in V in which both A and B, are diagonal, if and only if, the mapping $A^{-1}B$, possesses n-real eigenvalues."

Thus, if we take the pair $\gamma_{\alpha\beta}$, $\eta_{\alpha\beta}$, it suffices to prove that $\eta^{\alpha\varrho}\gamma_{\varrho\beta}$, has n-real eigenvalues. In what follows, for the sake of completeness, we give a proof of the entire statement that a positive definite matrix $\gamma_{\alpha\beta}$, can be diagonalized via the Lorentz group. **Theorem** Let γ be a positive definite $n \times n$ real matrix. Then, there exists a Lorentz matrix Λ, such that:

$$\Lambda^T \gamma \Lambda = \Delta \tag{A.1}$$

where Δ a diagonal matrix.

Note: Since $\Lambda^T = \eta\Lambda^{-1}\eta$, where η is the Minkowski metric, (A.1) may be written as

$$\Lambda^{-1}\eta\gamma\Lambda = \eta\Delta \tag{A.2}$$

In order to prove (A.2) it is useful to write it equivalently using the notation employed with linear mappings. To do that, we consider an n-dimensional real linear space V with basis $(e_1,\ e_2, \ldots,\ e_n)$. The scalar product in this space is defined as $<\ ,\ >\ :\ V \times V \rightarrow \Re$, with $< e_\alpha,\ e_\beta >= \eta_{\alpha\beta}$. The matrix $\eta\gamma$ defines a mapping $f:\ V \rightarrow V$ through the relation:

$$f(e_\alpha) = \sum_{\beta=1}^{n} (\eta\gamma)_{\alpha\beta} e_\beta$$

The following will prove useful later on:

1) If $M \subseteq V$ then $V = M \oplus M^{\perp}$ [23].

2) A mapping $f: \ V \to V$ is called self-dual, if

$< f(x),\ y >=< x,\ f(y) >$ for every $x,\ y\ \in\ V$. We may prove that the mapping f defined through the matrix $\eta\gamma$ is self-dual. Indeed:

$$\left.\begin{array}{c} < f(x),\ y >=< y,\ f(x) >= y^T\eta\eta\gamma x = y^T\gamma x \\ < x,\ f(y) >= x^T\eta\eta\gamma y = x^T\gamma y = y^T\gamma x \end{array}\right\}$$

$$\Rightarrow \quad < f(x),\ y >=< x,\ f(x) >$$

3) If $M \subseteq V$ is an invariant subspace of V with respect to a self-dual mapping f then $M^{\perp}$ is also an invariant subspace of V. Indeed, let $b \in M^{\perp}$ and $m \in M$. Since M is an invariant subspace, it follows that:

$$f(m) \in M \Rightarrow \quad < f(m),\ b >= 0 \Rightarrow \quad < m,\ f(b) >= 0\ \forall\ m \in M$$

$$\Rightarrow f(b) \in M^{\perp}$$

Equation (A.2) states the fact that there exists an orthonormal basis of V consisting of the eigenvalues of f. If (A.2) holds then the non-vanishing elements of the real diagonal matrix $\eta\Delta$ will be eigenvalues of $\eta\gamma$. Thus, we have to prove that the eigenvalues of $\eta\gamma$ are real. Indeed, the following theorem holds:

<u>Theorem</u>

If γ is a positive definite symmetric matrix, then $\eta\gamma$ has real eigenvalues.

<u>Proof</u>

Let $\lambda = \alpha + \beta j,\ \ \beta \neq 0$ a complex eigenvalue of $\eta\gamma$ and $u \neq 0$ the corresponding complex right eigenvector. Since η is invertible, there exists a $v = x + yj,\ \ x,\ y,\ \ \in \Re^n$ such that $u = \eta v$. We have:

$$\eta\gamma u = \lambda u \Leftrightarrow \eta\gamma\eta v = \lambda\eta v \Leftrightarrow$$

$$\eta\gamma\eta x = \alpha\eta x - \beta\eta y \tag{A.3}$$

$$\eta\gamma\eta y = \alpha\eta y + \beta\eta x \tag{A.4}$$

Equations (A.3), (A.4) imply respectively:

$$y^T\eta\gamma\eta x = \alpha < y,\ x > -\beta < y,\ y >$$

$$x^T\eta\gamma\eta y = \alpha < x,\ y > +\beta < x,\ x >$$

The last two equations have their left-hand sides equal (since $\eta\gamma\eta$ is symmetric), hence:

$$\beta(< x,\ x > + < y,\ y >) = 0 \Rightarrow < y,\ y >= - < x,\ x > \tag{A.5}$$

Since γ is positive definite, $\eta\gamma\eta$ is positive definite as well. Then:

$$x^T\eta\gamma\eta x \ \geq\ 0 \overset{(A.3)}{\Rightarrow} \alpha < x,\ x > -\beta < x,\ y > \ \geq\ 0 \tag{A.6}$$

$$\begin{gathered} y^T\eta\gamma\eta y \ \geq\ 0 \overset{(A.4)}{\Rightarrow} \alpha < y,\ y > +\beta < y,\ x > \ \geq\ 0 \overset{(A.5)}{\Rightarrow} \\ \alpha < x,\ x > -\beta < x,\ y > \ \leq\ 0 \end{gathered} \tag{A.7}$$

¿From (A.6), (A.7) we get:

$$\alpha < x,\ x >= \beta < x,\ y > \tag{A.8}$$

Through (A.5), (A.8), equations (A.3), (A.4) imply:

$$x^T \eta\gamma\eta x = 0$$

$$y^T \eta\gamma\eta y = 0$$

and, since $\eta\gamma\eta$ is positive definite we conclude that $x = 0$ and $y = 0$, i.e. $u = 0$, contradicting our initial assumption $u \neq 0$. Therefore β has to vanish and thus we have proved the reality of λ.

For the eigenvectors of $\eta\gamma$, we can prove that they have a non-zero norm. Indeed, let x be an eigenvector of $\eta\gamma$, i.e.

$$\eta\gamma x = \lambda x \Rightarrow \gamma x = \lambda\eta x \Rightarrow x^T\gamma x = \lambda x^T\eta x = \lambda < x,\ x >$$

Since γ is positive definite and $x \neq 0$ we have $x^T\gamma x > 0$, so that $< x,\ x > \neq 0$.

We are now in position to prove a spectral theorem for a mapping f with real eigenvalues.

Theorem

If $f : V \to V$ is a self-dual mapping with real eigenvalues, then V has an orthonormal basis consisting of the eigenvectors of f.

Proof

Let λ be an eigenvalue of f, u the corresponding eigenvector and $M = [u]$ the one-dimensional subspace spanned by u. Obviously, M is an invariant subspace of V with respect to f.

According to 1), we have $V = M \oplus M^{\perp}$. As implied by 2) and 3), $M^{\perp}$ is also an invariant subspace and thus f induces a self-dual mapping onto $M^{\perp}$. Hence, we can apply induction and show that:

$$V = M_1 \oplus M_2 \oplus \ldots \oplus M_n$$

where the M_α are one-dimensional invariant subspaces orthogonal to each other. Since u_α is an eigenvector of $\eta\gamma$, it holds that $< u,\ u > \neq 0$, as proved above. We can thus promote the orthogonal basis to an orthonormal set $(\hat{u}_1,\ \hat{u}_2, \ldots,\ \hat{u}_n)$. The transformation connecting this orthonormal basis to the initial orthonormal basis $(e_1,\ e_2, \ldots,\ e_n)$ is the matrix Λ sought for in the first theorem, relations (A.2) and (A.1).

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